Give or Take: Non-Determinism, Linear Logic and Session Types

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December 17, 2018

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Example: dramatis personæ

👩🏻 Mary

👦🏾 John

🏠 Pâtisserie

🍰 Cake

🙌 No Cake

Example: buying cake at the pâtisserie

$$\nu x.(\quad x(y).\text{👦🏾} \quad | \quad x(z).\text{👩🏻} \quad | \quad x\langle \text{🍰} \rangle . x \langle \text{🙌} \rangle . \text{🏡} \quad)$$

$$(\ \text{👦🏾}\{\text{🍰}/y\} \mid \text{👩🏻}\{\text{🙌}/z\} \mid \text{🏡}\ ) \qquad\qquad (\ \text{👦🏾}\{\text{🙌}/y\} \mid \text{👩🏻}\{\text{🍰}/z\} \mid \text{🏡}\ )$$

The π-calculus: syntax

Term P, Q, R

$$
\begin{array}{rll}
::= & \nu x.P & \text{create a new channel } x\text{, then run P} \\
\mid & x(y).P & \text{receive } y \text{ over } x\text{, then run P} \\
\mid & x\langle y\rangle.P & \text{send } y \text{ over } x\text{, then run P} \\
\mid & (P \mid Q) & \text{run P and Q in parallel} \\
\mid & 0 & \text{halt}
\end{array}
$$

Example: buying cake at the pâtisserie

$\nu x.($ $x(y).$👦🏾 $\mid$ $x(z).$👩🏻 $\mid$ $x\langle$🍰$\rangle.x\langle$🙌$\rangle.$🏡 $)$

$($ 👦🏾$\{$🍰$/y\}$ $\mid$ 👩🏻$\{$🙌$/z\}$ $\mid$ 🏡 $)$

$($ 👦🏾$\{$🙌$/y\}$ $\mid$ 👩🏻$\{$🍰$/z\}$ $\mid$ 🏡 $)$

Pitfall: illegal resale of cake

💰 Money

$$\nu x.(\quad x(\text{💰}).x\langle\text{🍰}\rangle.\text{👦🏾} \quad \mid \quad x(\text{🍰}).x\langle\text{💰}\rangle.\text{👩🏻} \quad)$$

Session types to the rescue

$$\text{Type } A, B ::= \alpha \mid \alpha^\perp \mid A \otimes B \mid A \parr B \mid \ldots$$

🎂$^\perp$	channel from which you **get cake**
🎂	channel in which you **put cake**
💰$^\perp$	channel from which you **get money**
💰	channel in which you **put money**
$A \otimes B$	pair of **independent** channels
$A \parr B$	pair of possibly **dependent** channels

Session types to the rescue

$$\text{Type } A, B ::= \alpha \mid \alpha^\perp \mid A \otimes B \mid A \parr B \mid \ldots$$

$$\frac{P \vdash \Gamma, y : A \qquad Q \vdash \Delta, x : B}{\nu y.x\langle y\rangle.(P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B}\otimes$$

$$\frac{P \vdash \Gamma, y : A, x : B}{x(y).P \vdash \Gamma, x : A \parr B}\parr$$

Session types to the rescue

$$\text{Type } A, B ::= \alpha \mid \alpha^{\perp} \mid A \otimes B \mid A \parr B \mid \ldots$$

$$\frac{P \vdash \Gamma, x : A \qquad Q \vdash \Delta, x : A^{\perp}}{\nu x.(P \mid Q) \vdash \Gamma, \Delta}\ \textsc{Cut}$$

Session types to the rescue

Type of x used by 👦🏾	Type of x used by 👩🏻
💰$^{\perp}$ ⅋ 🎂	💰 $\otimes$ 🎂$^{\perp}$
💰$^{\perp}$ $\otimes$ 🎂	💰 ⅋ 🎂$^{\perp}$

Session types to the rescue

$$\nu x.(\quad x(y).\text{👦🏾} \quad \mid \quad \nu w.x\langle w\rangle.(\; w \leftrightarrow \text{💰} \mid x(z).\text{👩🏻} \;) \quad)$$

Session types to the rescue

$$x[y].P ::= \nu y.x\langle y\rangle.P$$

$$\nu x.(\quad x(y).\text{👦🏾} \quad | \quad x[w].(\, w \leftrightarrow \text{💰} \;|\; x(z).\text{👩🏻} \,) \quad)$$

Example: buying cake at the pâtisserie

$\nu x.($ $x(y).$👦🏾 $|$ $x(z).$👩🏻 $|$ $x[$🍰$].x[$🙌$].$🏡 $)$

(👦🏾$\{$🍰$/y\}$ $|$ 👩🏻$\{$🙌$/z\}$ $|$ 🏡)

(👦🏾$\{$🙌$/y\}$ $|$ 👩🏻$\{$🍰$/z\}$ $|$ 🏡)

Non-Deterministic Classical Processes

$$\frac{P \vdash \Gamma, x : A \qquad Q \vdash \Delta, x : A^{\perp}}{\nu x.(P \mid Q) \vdash \Gamma, \Delta}\ \textsc{Cut}$$

Non-Deterministic Classical Processes

$$👦🏾 \vdash \Gamma, y : 🎂^{\perp} \qquad 👩🏻 \vdash \Delta, z : 🎂^{\perp} \qquad 🏡 \vdash \Theta, 🍰 : 🎂, 🙌 : 🎂$$

$$\vdots$$

$$\nu x.(\,(\,x(y).👦🏾 \mid x(z).👩🏻\,) \mid x[🍰].x[🙌].🏡\,) \vdash \Gamma, \Delta, \Theta$$

$$(\,👦🏾\{🍰/y\} \mid 👩🏻\{🙌/z\} \mid 🏡\,) \qquad (\,👦🏾\{🙌/y\} \mid 👩🏻\{🍰/z\} \mid 🏡\,)$$

Non-Deterministic Classical Processes

$$\text{👦🏾} \vdash \Gamma, y : \text{🎂}^{\perp} \qquad \text{👩🏻} \vdash \Delta, z : \text{🎂}^{\perp} \qquad \text{🏠} \vdash \Theta, \text{🍰} : \text{🎂}, \text{🙌} : \text{🎂}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\dfrac{(\ x(y).\text{👦🏾} \mid x(z).\text{👩🏻}\ ) \vdash \Gamma, \Delta, x : ??? \qquad x[\text{🍰}].x[\text{🙌}].\text{🏠} \vdash \Theta, x : ???}{\nu x.(\ (\ x(y).\text{👦🏾} \mid x(z).\text{👩🏻}\ ) \mid x[\text{🍰}].x[\text{🙌}].\text{🏠}\ ) \vdash \Gamma, \Delta, \Theta}\ \textsc{Cut}$$

$$(\ \text{👦🏾}\{\text{🍰}/y\} \mid \text{👩🏻}\{\text{🙌}/z\} \mid \text{🏠}\ ) \qquad\qquad (\ \text{👦🏾}\{\text{🙌}/y\} \mid \text{👩🏻}\{\text{🍰}/z\} \mid \text{🏠}\ )$$

Non-Deterministic Classical Processes

$$\text{👦🏾} \vdash \Gamma, y : \text{🎂}^{\perp} \qquad \text{👩🏻} \vdash \Delta, z : \text{🎂}^{\perp} \qquad\qquad \text{🏡} \vdash \Theta, \text{🍰} : \text{🎂}, \text{🙌} : \text{🎂}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\frac{(\, x(y).\text{👦🏾} \mid x(z).\text{👩🏻} \,) \vdash \Gamma, \Delta, x : !_2\text{🎂}^{\perp} \qquad x[\text{🍰}].x[\text{🙌}].\text{🏡} \vdash \Theta, x : ?_2\text{🎂}}{\nu x.(\, (\, x(y).\text{👦🏾} \mid x(z).\text{👩🏻} \,) \mid x[\text{🍰}].x[\text{🙌}].\text{🏡} \,) \vdash \Gamma, \Delta, \Theta}\ \textsc{Cut}$$

$$(\, \text{👦🏾}\{\text{🍰}/y\} \mid \text{👩🏻}\{\text{🙌}/z\} \mid \text{🏡} \,) \qquad\qquad (\, \text{👦🏾}\{\text{🙌}/y\} \mid \text{👩🏻}\{\text{🍰}/z\} \mid \text{🏡} \,)$$

$$\text{Type } A, B ::= \cdots \mid ?_n A \mid !_n A$$

$$\cfrac{\cfrac{\text{👦🏾} \vdash \Gamma, y : \text{🎂}^{\perp}}{x(y).\text{👦🏾} \vdash \Gamma, x : !_1\text{🎂}^{\perp}}\ !_1 \qquad \cfrac{\text{👩🏻} \vdash \Delta, z : \text{🎂}^{\perp}}{x(z).\text{👩🏻} \vdash \Delta, x : !_1\text{🎂}^{\perp}}\ !_1}{(\ x(y).\text{👦🏾} \ \mid\ x(z).\text{👩🏻}\ ) \vdash \Gamma, \Delta, x : !_2\text{🎂}^{\perp}}\ \textsc{Pool}$$

Non-Deterministic Classical Proccesses

$$\text{Type } A, B ::= \cdots \mid ?_n A \mid !_n A$$

$$
\dfrac{\dfrac{\dfrac{\text{🏡} \vdash \Theta, \text{🍰} : \text{🎂}, \text{🙌} : \text{🎂}}{x'[\text{🙌}].\text{🏡} \vdash \Theta, \text{🍰} : \text{🎂}, x' : ?_1\text{🎂}}\ ?_1}{x[\text{🍰}].x'[\text{🙌}].\text{🏡} \vdash \Theta, x : ?_1\text{🎂}, x' : ?_1\text{🎂}}\ ?_1}{x[\text{🍰}].x[\text{🙌}].\text{🏡} \vdash \Theta, x : ?_2\text{🎂}}\ \textsc{Contract}
$$

Non-Deterministic Classical Processes

$$\text{Type } A, B ::= \cdots \mid ?_n A \mid !_n A$$

$$\frac{P \vdash \Gamma, y : A}{x[y].P \vdash \Gamma, x : ?_1 A}\ ?_1 \qquad \frac{P \vdash \Gamma, y : A}{x(y).P \vdash \Gamma, x : !_1 A}\ !_1$$

$$\frac{P \vdash \Gamma, x : ?_m A, y : ?_n A}{P\{x/y\} \vdash \Gamma, x : ?_{m+n} A}\ \textsc{Contract}$$

$$\frac{P \vdash \Gamma, x : !_m A \qquad Q \vdash \Delta, x : !_n A}{(P \mid Q) \vdash \Gamma, \Delta, x : !_{m+n} A}\ \textsc{Pool}$$

Cut-elimination/Communication

$$\frac{P \vdash \Gamma, x : ?_n A \qquad Q \vdash \Delta, x : !_n A^\perp}{\nu x.(P \mid Q) \vdash \Gamma, \Delta}\ \textsc{Cut}$$

Cut-elimination/Communication

$$\frac{P \vdash \Gamma, x : ?_n A}{x \uparrow y_1 \cdots y_n.P \vdash \Gamma, y_1 : A \cdots y_n : A}\ \textsc{Exp}$$

$$\frac{P \vdash \Gamma, y_1 : A \cdots y_n : A \qquad Q \vdash \Delta, x : !_n A^{\perp}}{x \downarrow y_1 \cdots y_n.Q \vdash \Delta}\ \textsc{Int}$$

Non-Deterministic Classical Proccesses

$$\text{Type } A, B ::= \cdots \mid ?_n A \mid !_n A$$

$$\dfrac{\dfrac{\dfrac{\text{🏡} \vdash \Theta, \text{🍰} : \text{🎂}, \text{🙌} : \text{🎂}}{x'[\text{🙌}].\text{🏡} \vdash \Theta, \text{🍰} : \text{🎂}, x' : ?_1\text{🎂}}\;?_1}{x[\text{🍰}].x'[\text{🙌}].\text{🏡} \vdash \Theta, x : ?_1\text{🎂}, x' : ?_1\text{🎂}}\;?_1}{x[\text{🍰}].x[\text{🙌}].\text{🏡} \vdash \Theta, x : ?_2\text{🎂}}\;\textsc{Contract}$$

Non-Deterministic Classical Proccesses

Type $A, B ::= \cdots \mid ?_n A \mid !_n A$

$$\dfrac{\dfrac{\text{👦🏾} \vdash \Gamma, y : \text{🎂}^{\perp}}{x(y).\text{👦🏾} \vdash \Gamma, x : !_1 \text{🎂}^{\perp}}\;!_1 \qquad \dfrac{\text{👩🏻} \vdash \Delta, z : \text{🎂}^{\perp}}{x(z).\text{👩🏻} \vdash \Delta, x : !_1 \text{🎂}^{\perp}}\;!_1}{(\, x(y).\text{👦🏾} \mid x(z).\text{👩🏻} \,) \vdash \Gamma, \Delta, x : !_2 \text{🎂}^{\perp}}\;\textsc{Pool}$$

Cut-elimination/Communication

$$
\dfrac{\dfrac{\text{👦🏾}\{\text{🍰}/y\} \vdash \Gamma, \text{🍰} : \text{🎂}^{\perp} \qquad \text{🏠} \vdash \Theta, \text{🍰} : \text{🎂}, \text{🙌} : \text{🎂}}{\nu\text{🍰}.(\,\text{👦🏾}\{\text{🍰}/y\} \mid \text{🏠}\,) \vdash \Gamma, \Theta, \text{🙌} : \text{🎂}}\,\textsc{Cut} \qquad \text{👩🏻}\{\text{🙌}/z\} \vdash \Delta, \text{🙌} : \text{🎂}^{\perp}}{\nu\text{🙌}.(\,\text{👩🏻}\{\text{🙌}/z\} \mid \nu\text{🍰}.(\,\text{👦🏾}\{\text{🍰}/y\} \mid \text{🏠}\,)\,) \vdash \Gamma, \Theta}\,\textsc{Cut}
$$

$$
\dfrac{\dfrac{\text{👩🏻}\{\text{🍰}/y\} \vdash \Gamma, \text{🍰} : \text{🎂}^{\perp} \qquad \text{🏠} \vdash \Theta, \text{🍰} : \text{🎂}, \text{🙌} : \text{🎂}}{\nu\text{🍰}.(\,\text{👩🏻}\{\text{🍰}/y\} \mid \text{🏠}\,) \vdash \Gamma, \Theta, \text{🙌} : \text{🎂}}\,\textsc{Cut} \qquad \text{👦🏾}\{\text{🙌}/z\} \vdash \Delta, \text{🙌} : \text{🎂}^{\perp}}{\nu\text{🙌}.(\,\text{👦🏾}\{\text{🙌}/z\} \mid \nu\text{🍰}.(\,\text{👩🏻}\{\text{🍰}/y\} \mid \text{🏠}\,)\,) \vdash \Gamma, \Theta}\,\textsc{Cut}
$$

Cut-elimination/Communication

$$\dfrac{P \vdash \Gamma, x : ?_n A \qquad Q \vdash \Delta, x : !_n A^\perp}{\nu x.(P \mid Q) \vdash \Gamma, \Delta}\ \textsc{Cut}$$

$$\Downarrow$$

$$\dfrac{\dfrac{P \vdash \Gamma, x : ?_n A}{x \uparrow y_1 \cdots y_n.P \vdash \Gamma, y_1 : A \cdots y_n : A}\ \textsc{Exp} \qquad Q \vdash \Delta, x : !_n A^\perp}{x \downarrow y_1 \cdots y_n.(x \uparrow y_1 \cdots y_n.P \mid Q) \vdash \Delta}\ \textsc{Int}$$

Related and Future Work

I should mention...

- type system implemented in Agda;
- cut-elimination procedure implemented in Agda;

And I'm looking into...

- extension with resource quantifiers;
- embedding CP back into $\mathrm{CP}_{\mathrm{NP}}$.

Non-Deterministic Classical Processes

$$\text{Type } A, B ::= \cdots \mid ?_n A \mid !_n A$$

$$\frac{P \vdash \Gamma, y : A}{x[y].P \vdash \Gamma, x : ?_1 A}\ ?_1 \qquad \frac{P \vdash \Gamma, y : A}{x(y).P \vdash \Gamma, x : !_1 A}\ !_1$$

$$\frac{P \vdash \Gamma, x : ?_m A, y : ?_n A}{P\{x/y\} \vdash \Gamma, x : ?_{m+n} A}\ \textsc{Contract}$$

$$\frac{P \vdash \Gamma, x : !_m A \qquad Q \vdash \Delta, x : !_n A}{(P \mid Q) \vdash \Gamma, \Delta, x : !_{m+n} A}\ \textsc{Pool}$$

Resource Quantifiers

$$\text{Type } A, B ::= \ldots \mid \forall n.A \mid \exists n.A$$

$$!A := \forall n.!_n A$$

$$?A := \exists n.?_n A$$

$$\text{Server}(A) := \forall n.?_n A$$

$$\text{Client}(A) := \exists n.!_n A$$