

Give or Take: Non-Determinism, Linear Logic and Session Types

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Example: dramatis personæ



Mary



John



Pâtisserie



Cake



No Cake

Example: buying cake at the pâtisserie

$\nu x.(x(y).\text{👤} \mid x(z).\text{👩} \mid x(\text{🍰}).x(\text{👏}).\text{🏠})$

$(\text{👤}\{\text{🍰}/y\} \mid \text{👩}\{\text{👏}/z\} \mid \text{🏠})$

$(\text{👤}\{\text{👏}/y\} \mid \text{👩}\{\text{🍰}/z\} \mid \text{🏠})$

The π -calculus: syntax

Term P, Q, R

- $::= \nu x.P$ create a new channel x , then run P
- | $x(y).P$ receive y over x , then run P
- | $x\langle y \rangle.P$ send y over x , then run P
- | $(P \mid Q)$ run P and Q in parallel
- | 0 halt

Example: buying cake at the pâtisserie

$\nu x.(x(y).\text{👤} \mid x(z).\text{👩} \mid x(\text{🍰}).x(\text{👏}).\text{🏠})$

$(\text{👤}\{\text{🍰}/y\} \mid \text{👩}\{\text{👏}/z\} \mid \text{🏠})$

$(\text{👤}\{\text{👏}/y\} \mid \text{👩}\{\text{🍰}/z\} \mid \text{🏠})$

Pitfall: illegal resale of cake



Money

$\nu x. (x(\text{money bag}).x(\text{cake}).\text{man} \mid x(\text{cake}).x(\text{money bag}).\text{woman})$

Session types to the rescue

Type $A, B ::= \alpha \mid \alpha^\perp \mid A \otimes B \mid A \wp B \mid \dots$



channel from which you **get cake**



channel in which you **put cake**



channel from which you **get money**



channel in which you **put money**

$A \otimes B$

pair of **independent** channels

$A \wp B$

pair of possibly **dependent** channels

Session types to the rescue

Type $A, B ::= \alpha \mid \alpha^\perp \mid A \otimes B \mid A \wp B \mid \dots$

$$\frac{P \vdash \Gamma, y : A \quad Q \vdash \Delta, x : B}{\nu y. x \langle y \rangle. (P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B} \otimes$$

$$\frac{P \vdash \Gamma, y : A, x : B}{x(y). P \vdash \Gamma, x : A \wp B} \wp$$

Session types to the rescue

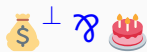
Type $A, B ::= \alpha \mid \alpha^\perp \mid A \otimes B \mid A \wp B \mid \dots$

$$\frac{P \vdash \Gamma, x : A \quad Q \vdash \Delta, x : A^\perp}{\nu x. (P \mid Q) \vdash \Gamma, \Delta} \text{CUT}$$

Session types to the rescue

Type of x used by 

Type of x used by 



Session types to the rescue

$$\nu x. (x(y). \text{👤} \mid \nu w. x \langle w \rangle. (w \leftrightarrow \text{💰} \mid x(z). \text{👩}))$$

Session types to the rescue

$$x[y].P ::= \nu y.x\langle y \rangle.P$$

$$\nu x.(x(y).\text{👤} \mid x[w].(w \leftrightarrow \text{💰} \mid x(z).\text{👩}))$$

Example: buying cake at the pâtisserie

$\nu x.(x(y).\text{👤} \mid x(z).\text{👩} \mid x[\text{🍰}].x[\text{👏👏}].\text{🏠})$

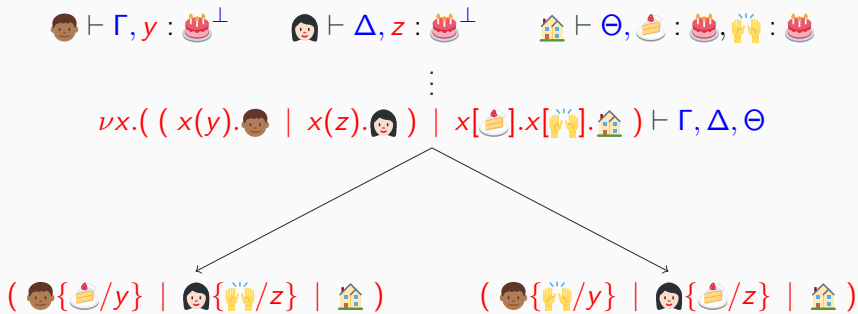
$(\text{👤}\{\text{🍰}/y\} \mid \text{👩}\{\text{👏👏}/z\} \mid \text{🏠})$

$(\text{👤}\{\text{👏👏}/y\} \mid \text{👩}\{\text{🍰}/z\} \mid \text{🏠})$

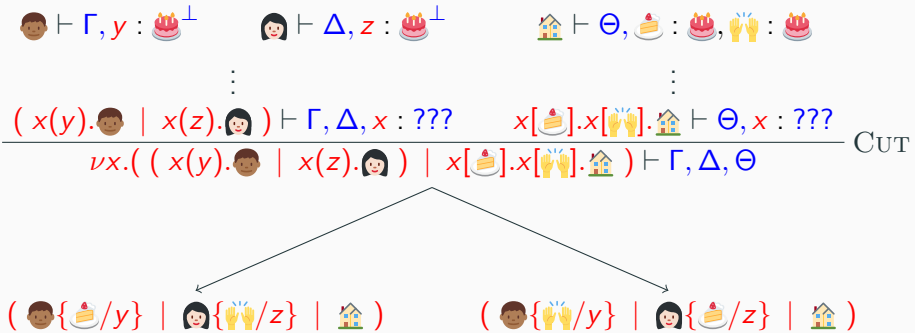
Non-Deterministic Classical Processes

$$\frac{P \vdash \Gamma, x : A \quad Q \vdash \Delta, x : A^\perp}{\nu x. (P \mid Q) \vdash \Gamma, \Delta} \text{CUT}$$

Non-Deterministic Classical Processes



Non-Deterministic Classical Processes



Non-Deterministic Classical Processes

$$\frac{\begin{array}{c} \text{👤} \vdash \Gamma, y : \text{🍰}^\perp \\ \vdots \\ (x(y).\text{👤} \mid x(z).\text{👩}) \vdash \Gamma, \Delta, x : !_2 \text{🍰}^\perp \end{array} \quad \begin{array}{c} \text{👩} \vdash \Delta, z : \text{🍰}^\perp \\ \vdots \\ x[\text{🍰}].x[\text{👏}].\text{🏠} \vdash \Theta, x : ?_2 \text{🍰} \end{array}}{\nu x. ((x(y).\text{👤} \mid x(z).\text{👩}) \mid x[\text{🍰}].x[\text{👏}].\text{🏠}) \vdash \Gamma, \Delta, \Theta} \text{CUT}$$
$$\begin{array}{c} \swarrow \\ (\text{👤} \{ \text{🍰} / y \} \mid \text{👩} \{ \text{👏} / z \} \mid \text{🏠}) \end{array} \quad \begin{array}{c} \searrow \\ (\text{👤} \{ \text{👏} / y \} \mid \text{👩} \{ \text{🍰} / z \} \mid \text{🏠}) \end{array}$$

Non-Deterministic Classical Processes

Type $A, B ::= \dots \mid ?_n A \mid !_n A$

$$\frac{\frac{\text{👤} \vdash \Gamma, y : \text{🍰}^\perp}{x(y).\text{👤} \vdash \Gamma, x : !_1 \text{🍰}^\perp} !_1 \quad \frac{\text{👩} \vdash \Delta, z : \text{🍰}^\perp}{x(z).\text{👩} \vdash \Delta, x : !_1 \text{🍰}^\perp} !_1}{(x(y).\text{👤} \mid x(z).\text{👩}) \vdash \Gamma, \Delta, x : !_2 \text{🍰}^\perp} \text{POOL}}$$

Non-Deterministic Classical Processes

Type $A, B ::= \dots \mid ?_n A \mid !_n A$

$$\frac{\frac{\frac{\text{house} \vdash \Theta, \text{cake} : \text{cake}, \text{clapping} : \text{cake}}{x'[\text{clapping}]. \text{house} \vdash \Theta, \text{cake} : \text{cake}, x' : ?_1 \text{cake}}{?_1}}{x[\text{cake}]. x'[\text{clapping}]. \text{house} \vdash \Theta, x : ?_1 \text{cake}, x' : ?_1 \text{cake}}{?_1}}{x[\text{cake}]. x[\text{clapping}]. \text{house} \vdash \Theta, x : ?_2 \text{cake}}{\text{CONTRACT}}$$

Non-Deterministic Classical Processes

Type $A, B ::= \dots \mid ?_n A \mid !_n A$

$$\frac{P \vdash \Gamma, y : A}{x[y].P \vdash \Gamma, x : ?_1 A} ?_1 \quad \frac{P \vdash \Gamma, y : A}{x(y).P \vdash \Gamma, x : !_1 A} !_1$$

$$\frac{P \vdash \Gamma, x : ?_m A, y : ?_n A}{P\{x/y\} \vdash \Gamma, x : ?_{m+n} A} \text{CONTRACT}$$

$$\frac{P \vdash \Gamma, x : !_m A \quad Q \vdash \Delta, x : !_n A}{(P \mid Q) \vdash \Gamma, \Delta, x : !_{m+n} A} \text{POOL}$$

$$\frac{P \vdash \Gamma, x : ?_n A \quad Q \vdash \Delta, x : !_n A^\perp}{\nu x. (P \mid Q) \vdash \Gamma, \Delta} \text{CUT}$$

$$\frac{P \vdash \Gamma, x : ?_n A}{x \uparrow y_1 \cdots y_n. P \vdash \Gamma, y_1 : A \cdots y_n : A} \text{EXP}$$

$$\frac{P \vdash \Gamma, y_1 : A \cdots y_n : A \quad Q \vdash \Delta, x : !_n A^\perp}{x \downarrow y_1 \cdots y_n. Q \vdash \Delta} \text{INT}$$

Non-Deterministic Classical Processes

Type $A, B ::= \dots \mid ?_n A \mid !_n A$

$$\frac{\frac{\frac{\text{house} \vdash \Theta, \text{cake} : \text{cake}, \text{clapping} : \text{cake}}{x'[\text{clapping}]. \text{house} \vdash \Theta, \text{cake} : \text{cake}, x' : ?_1 \text{cake}}{?_1}}{x[\text{cake}]. x'[\text{clapping}]. \text{house} \vdash \Theta, x : ?_1 \text{cake}, x' : ?_1 \text{cake}}{?_1}}{x[\text{cake}]. x[\text{clapping}]. \text{house} \vdash \Theta, x : ?_2 \text{cake}}{\text{CONTRACT}}$$

Non-Deterministic Classical Processes

Type $A, B ::= \dots \mid ?_n A \mid !_n A$

$$\frac{\frac{\text{👤} \vdash \Gamma, y : \text{🍰}^\perp}{x(y).\text{👤} \vdash \Gamma, x : !_1 \text{🍰}^\perp} !_1 \quad \frac{\text{👩} \vdash \Delta, z : \text{🍰}^\perp}{x(z).\text{👩} \vdash \Delta, x : !_1 \text{🍰}^\perp} !_1}{(x(y).\text{👤} \mid x(z).\text{👩}) \vdash \Gamma, \Delta, x : !_2 \text{🍰}^\perp} \text{POOL}}$$

Cut-elimination/Communication

$$\frac{\frac{\text{👤}\{\text{🍰}/y\} \vdash \Gamma, \text{🍰} : \text{🎂}^\perp \quad \text{🏠} \vdash \Theta, \text{🍰} : \text{🎂}, \text{👏} : \text{🎂}}{\nu \text{🍰} . (\text{👤}\{\text{🍰}/y\} \mid \text{🏠}) \vdash \Gamma, \Theta, \text{👏} : \text{🎂}} \text{CUT} \quad \text{👩}\{\text{👏}/z\} \vdash \Delta, \text{👏} : \text{🎂}^\perp}{\nu \text{👏} . (\text{👩}\{\text{👏}/z\} \mid \nu \text{🍰} . (\text{👤}\{\text{🍰}/y\} \mid \text{🏠})) \vdash \Gamma, \Theta} \text{CUT}$$

$$\frac{\frac{\text{👩}\{\text{🍰}/y\} \vdash \Gamma, \text{🍰} : \text{🎂}^\perp \quad \text{🏠} \vdash \Theta, \text{🍰} : \text{🎂}, \text{👏} : \text{🎂}}{\nu \text{🍰} . (\text{👩}\{\text{🍰}/y\} \mid \text{🏠}) \vdash \Gamma, \Theta, \text{👏} : \text{🎂}} \text{CUT} \quad \text{👤}\{\text{👏}/z\} \vdash \Delta, \text{👏} : \text{🎂}^\perp}{\nu \text{👏} . (\text{👤}\{\text{👏}/z\} \mid \nu \text{🍰} . (\text{👩}\{\text{🍰}/y\} \mid \text{🏠})) \vdash \Gamma, \Theta} \text{CUT}$$

Cut-elimination/Communication

$$\frac{P \vdash \Gamma, x : ?_n A \quad Q \vdash \Delta, x : !_n A^\perp}{\nu x.(P \mid Q) \vdash \Gamma, \Delta} \text{CUT}$$

↓

$$\frac{\frac{P \vdash \Gamma, x : ?_n A}{x \uparrow y_1 \cdots y_n. P \vdash \Gamma, y_1 : A \cdots y_n : A} \text{EXP} \quad Q \vdash \Delta, x : !_n A^\perp}{x \downarrow y_1 \cdots y_n.(x \uparrow y_1 \cdots y_n. P \mid Q) \vdash \Delta} \text{INT}$$

I should mention. . .

- type system implemented in Agda;
- cut-elimination procedure implemented in Agda;

And I'm looking into. . .

- extension with resource quantifiers;
- embedding CP back into CP_{NP} .

Non-Deterministic Classical Processes

Type $A, B ::= \dots \mid ?_n A \mid !_n A$

$$\frac{P \vdash \Gamma, y : A}{x[y].P \vdash \Gamma, x : ?_1 A} ?_1 \quad \frac{P \vdash \Gamma, y : A}{x(y).P \vdash \Gamma, x : !_1 A} !_1$$

$$\frac{P \vdash \Gamma, x : ?_m A, y : ?_n A}{P\{x/y\} \vdash \Gamma, x : ?_{m+n} A} \text{CONTRACT}$$

$$\frac{P \vdash \Gamma, x : !_m A \quad Q \vdash \Delta, x : !_n A}{(P \mid Q) \vdash \Gamma, \Delta, x : !_{m+n} A} \text{POOL}$$

Type $A, B ::= \dots \mid \forall n.A \mid \exists n.A$

$!A := \forall n.!_n A$

$?A := \exists n.?_n A$

$\text{Server}(A) := \forall n.?_n A$

$\text{Client}(A) := \exists n.!_n A$