Global Non-Determinism With Termination

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This is how I write π-calculus things...

Term P, Q, R

$::=$	$\nu x.P$	create a new channel x, then run P
$\mid$	$x[y].P$	send a channel y over x, then run P
$\mid$	$x(y).P$	receive a channel y over x, then run P
$\mid$	0	halt
$\mid$	$(P \mid Q)$	run P and Q in parallel
$\mid$	$x \leftrightarrow y$	forward all messages on x to y and vice versa

This is how I write π-calculus things...

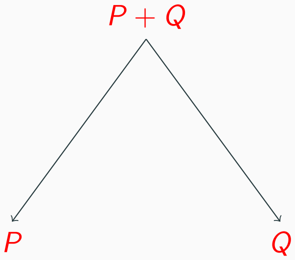
Term P, Q, R		
	$::= \ \dots$	
	$\mid x[\mathbf{inl}].P$	send a bit (**inl**) over x, then run P
	$\mid x[\mathbf{inr}].P$	send a bit (**inr**) over x, then run P
	$\mid \text{case } x \ \{P; Q\}$	receive a bit over x, then run P or Q

This is how I write π-calculus things...

Term P, Q, R

$$
\begin{aligned}
::= \; & \ldots \\
& \mid x[].P && \text{send a ping over } x\text{, then run P} \\
& \mid x().P && \text{receive a ping over } x\text{, then run P} \\
& \mid \mathsf{case}\ x\ \{\} && \text{loop}
\end{aligned}
$$

Local choice

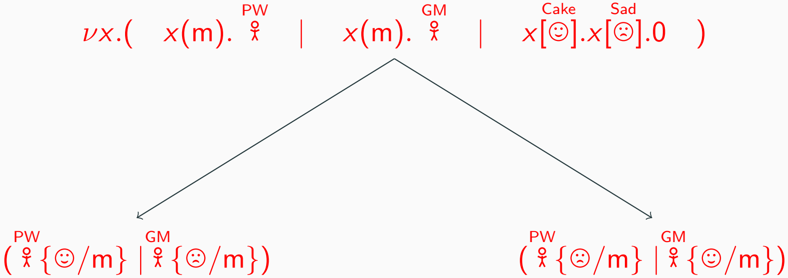


Non-determinism in the π-calculus (or "global" choice)

$$\nu x.(x(y).P \mid x(z).Q \mid x[a].x[b].0)$$

$$(P\{a/y\} \mid Q\{b/z\}) \qquad (P\{b/y\} \mid Q\{a/z\})$$

Example: a pâtisserie



Pros and cons

Local choice

Pros:

- It's pretty simple...
- We have typing rules...

Cons:

- Everything else!

Global choice

Pros:

- It's inherent in the π-calculus!

Cons:

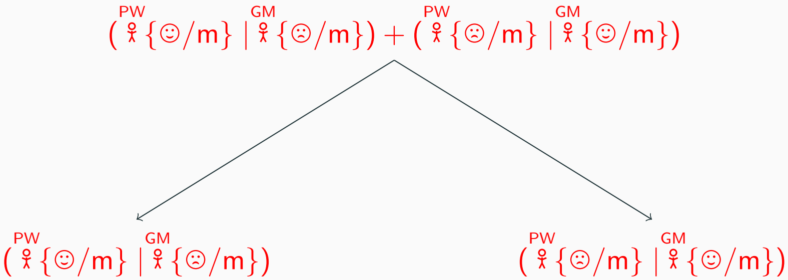
- We don't have typing rules...

Encoding global choice using local choice

$$(P\{a/y\} \mid Q\{b/z\}) + (P\{b/y\} \mid Q\{a/z\})$$

$$(P\{a/y\} \mid Q\{b/z\}) \qquad\qquad (P\{b/y\} \mid Q\{a/z\})$$

Encoding global choice using local choice



Local choice is not modular

If we extend…

$$\nu x.(x(y).P \mid x(z).Q \mid x[a].x[b].0)$$

…to…

$$\nu x.(x(y).P \mid x(z).Q \mid x(w).R \mid x[a].x[b].x[c].0)$$

Local choice is not modular

Then we must extend. . .

$$(P\{a/y\} \mid Q\{b/z\}) + (P\{b/y\} \mid Q\{a/z\})$$

. . . to. . .

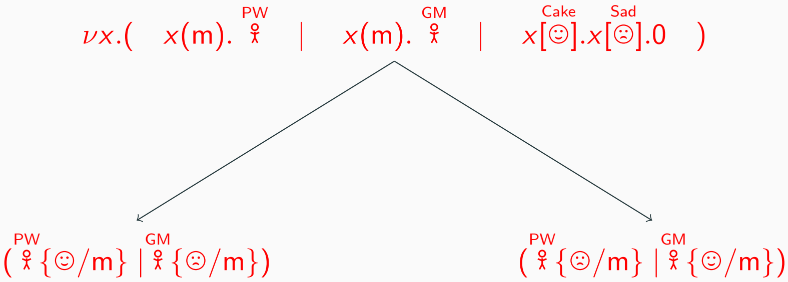
$$\begin{aligned}
&(P\{a/y\} \mid Q\{b/z\} \mid R\{c/w\}) &+\\
&(P\{b/y\} \mid Q\{a/z\} \mid R\{c/w\}) &+\\
&(P\{a/y\} \mid Q\{c/z\} \mid R\{b/w\}) &+\\
&(P\{b/y\} \mid Q\{c/z\} \mid R\{a/w\}) &+\\
&(P\{c/y\} \mid Q\{a/z\} \mid R\{b/w\}) &+\\
&(P\{c/y\} \mid Q\{b/z\} \mid R\{a/w\}) &
\end{aligned}$$

Encoding local choice using global choice

$$\nu x.(\text{case } x \, \{P; 0\} \mid \text{case } x \, \{Q; 0\} \mid x[\textbf{inl}].x[\textbf{inr}].0)$$

P $\qquad\qquad$ Q

Example: a pâtisserie



Classical Processes—Input and Output

$$\text{Type } A, B ::= \alpha \mid \alpha^\perp \mid A \otimes B \mid A \parr B$$

$$\dfrac{}{x \leftrightarrow y \vdash x : A, y : A^\perp}\ \textsc{Ax}$$

$$\dfrac{P \vdash \Gamma, x : A \qquad Q \vdash \Delta, x : A^\perp}{\nu x.(P \mid Q) \vdash \Gamma, \Delta}\ \textsc{Cut}$$

$$\dfrac{P \vdash \Gamma, y : A \qquad Q \vdash \Delta, x : B}{x[y].(P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B}\ \otimes$$

$$\dfrac{P \vdash \Gamma, y : A, x : B}{x(y).P \vdash \Gamma, x : A \parr B}\ \parr$$

Classical Processes—Input and Output

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$$\frac{P \vdash \Gamma, y : A, x : B}{x(y).P \vdash \Gamma, x : A \parr B}\ \parr$$

Classical Processes—Input and Output

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$$\frac{P \vdash \Gamma, y : A, x : B}{x(y).P \vdash \Gamma, x : A \parr B} \parr$$

Classical Processes—Choice and Selection

$$\text{Type } A, B ::= \cdots \mid A \oplus B \mid A \mathbin{\&} B$$

$$\frac{P \vdash \Gamma, x : A}{x[\textbf{inl}].P \vdash \Gamma, x : A \oplus B}\ \oplus_1 \qquad \frac{P \vdash \Gamma, x : B}{x[\textbf{inr}].P \vdash \Gamma, x : A \oplus B}\ \oplus_2$$

$$\frac{P \vdash \Gamma, x : A \qquad Q \vdash \Delta, x : B}{\text{case } x \,\{P; Q\} \vdash \Gamma, \Delta, x : A \mathbin{\&} B}\ \&$$

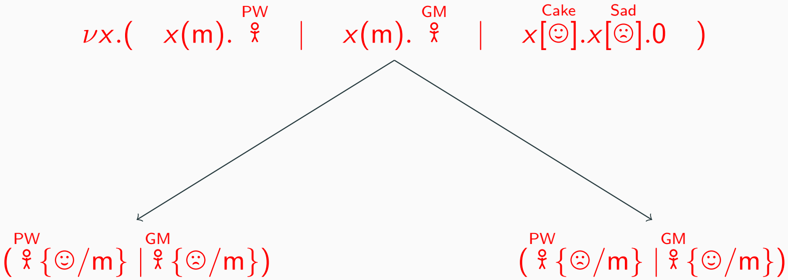
Classical Processes—Units

$$\text{Type } A, B ::= \cdots \mid \mathbf{1} \mid \bot \mid \mathbf{0} \mid \top$$

$$\frac{}{x[].0 \vdash x : \mathbf{1}}\ \mathbf{1} \qquad \frac{P \vdash \Gamma}{x().P \vdash \Gamma, x : \bot}\ \bot$$

(no rule for $\mathbf{0}$) $\qquad \dfrac{}{\text{case } x\ \{\} \vdash x : \top}\ \top$

Example: a pâtisserie



Classical Processes—Input and Output

$$\text{Type } A, B ::= \alpha \mid \alpha^\perp \mid A \otimes B \mid A \parr B$$

$$\frac{}{x{\leftrightarrow}y \vdash x : A, y : A^\perp}\ \textsc{Ax}$$

$$\frac{P \vdash \Gamma, x : A \qquad Q \vdash \Delta, x : A^\perp}{\nu x.(P \mid Q) \vdash \Gamma, \Delta}\ \textsc{Cut}$$

$$\frac{P \vdash \Gamma, y : A \qquad Q \vdash \Delta, x : B}{x[y].(P \mid Q) \vdash \Gamma, \Delta, x : A \otimes B}\ \otimes$$

$$\frac{P \vdash \Gamma, y : A, x : B}{x(y).P \vdash \Gamma, x : A \parr B}\ \parr$$

Non-Deterministic Classical Processes

$$\text{Type } A, B ::= \cdots \mid ?_n A \mid !_n A$$

$$\text{Term } P, Q, R ::= \cdots \mid ?x[y].P \mid !x(y).P$$

$$\frac{P \vdash \Gamma, y : A}{?x[y].P \vdash \Gamma, x : ?_1 A}\ ?_1 \qquad \frac{P \vdash \Gamma, y : A}{!x(y).P \vdash \Gamma, x : !_1 A}\ !_1$$

$$\frac{P \vdash \Gamma, x : ?_m A, y : ?_n A}{P\{x/y\} \vdash \Gamma, x : ?_{m+n} A}\ \textsc{Contract}$$

$$\frac{P \vdash \Gamma, x : !_m A \qquad Q \vdash \Delta, x : !_n A}{(P \mid Q) \vdash \Gamma, \Delta, x : !_{m+n} A}\ \textsc{Pool}$$

Example: a pâtisserie

$$
\dfrac{
  \dfrac{
    \dfrac{P \vdash \Gamma, y : A^\perp}{!x(y).P \vdash \Gamma, x : !_1 A^\perp}\ !
    \qquad
    \dfrac{Q \vdash \Delta, z : A^\perp}{!x(z).Q \vdash \Delta, x : !_1 A^\perp}\ !
  }{(!x(y).P \mid !x(z).Q) \vdash \Gamma, \Delta, x : !_2 A^\perp}\ \textsc{Pool}
  \qquad
  \dfrac{
    \dfrac{R \vdash \Theta, a : A, b : A}{?x[a].?w[b].R \vdash \Theta, x : ?_1 A, w : ?_1 A}\ ?(2)
  }{?x[a].?x[b].R \vdash \Theta, x : ?_2 A}\ \textsc{Cont}
}{\nu x.(!x(y).P \mid !x(z).Q \mid ?x[a].?x[b].R) \vdash \Gamma, \Delta, \Theta}\ \textsc{Cut}
$$

Example: a pâtisserie

$$
\dfrac{
\dfrac{
\dfrac{\overset{\text{PW}}{\text{🯅}} \vdash \Gamma, y : A^{\perp}}{!x(y).\ \overset{\text{PW}}{\text{🯅}} \vdash \Gamma, x : !_1 A^{\perp}}\ !
\qquad
\dfrac{\overset{\text{GM}}{\text{🯅}} \vdash \Delta, z : A^{\perp}}{!x(z).\ \overset{\text{GM}}{\text{🯅}} \vdash \Delta, x : !_1 A^{\perp}}\ !
}{(!x(y).\ \overset{\text{PW}}{\text{🯅}} \mid !x(z).\ \overset{\text{GM}}{\text{🯅}}) \vdash \Gamma, \Delta, x : !_2 A^{\perp}}\ \textsc{Pool}
\qquad
\dfrac{
\dfrac{0 \vdash \Theta, a : A, b : A}{?x[\text{☺}].?w[\text{☹}].0 \vdash \Theta, x : ?_1 A, w : ?_1 A}\ ?(2)
}{?x[\text{☺}].?x[\text{☹}].0 \vdash \Theta, x : ?_2 A}\ \textsc{Cont}
}{\nu x.(!x(y).\ \overset{\text{PW}}{\text{🯅}} \mid !x(z).\ \overset{\text{GM}}{\text{🯅}} \mid ?x[\text{☺}].?x[\text{☹}].0) \vdash \Gamma, \Delta, \Theta}\ \textsc{Cut}
$$

Cut-elimination procedure

$$\frac{P \vdash \Gamma, x : ?_n A \qquad Q \vdash \Delta, x : !_n A^\perp}{\nu x.(P \mid Q) \vdash \Gamma, \Delta}\ \textsc{Cut}$$

Cut-elimination procedure

$$\frac{P \vdash \Gamma, x : ?_n A}{x \uparrow y_1 \cdots y_n.P \vdash \Gamma, y_1 : A \cdots y_n : A}\ \textsc{Exp}$$

$$\frac{P \vdash \Gamma, y_1 : A \cdots y_n : A \qquad Q \vdash \Delta, x : !_n A^{\perp}}{x \downarrow y_1 \cdots y_n.Q \vdash \Delta}\ \textsc{Int}$$

Cut-elimination procedure

$$\dfrac{P \vdash \Gamma, x : ?_n A \qquad Q \vdash \Delta, x : !_n A^{\perp}}{\nu x.(P \mid Q) \vdash \Gamma, \Delta}\ \textsc{Cut}$$

$$\Downarrow$$

$$\dfrac{\dfrac{P \vdash \Gamma, x : ?_n A}{x \uparrow y_1 \cdots y_n.P \vdash \Gamma, y_1 : A \cdots y_n : A}\ \textsc{Exp} \qquad Q \vdash \Delta, x : !_n A^{\perp}}{x \downarrow y_1 \cdots y_n.(x \uparrow y_1 \cdots y_n.P \mid Q) \vdash \Delta}\ \textsc{Int}$$

Non-determinism

$$\dfrac{\dfrac{P \vdash \Gamma, x : ?_n A}{x \uparrow y_1 \cdots y_n.P \vdash \Gamma, y_1 : A \cdots y_n : A}\ \textsc{Exp}}{x \uparrow \mathit{shuffle}(y_1 \cdots y_n).P \vdash \Gamma, y_1 : A \cdots y_n : A}\ \mathrm{ND}$$

$$\vdots$$

Non-Deterministic Classical Processes

$$\text{Type } A, B ::= \cdots \mid ?_n A \mid !_n A$$

$$\text{Term } P, Q, R ::= \cdots \mid ?x[y].P \mid !x(y).P$$

$$\frac{P \vdash \Gamma, y : A}{?x[y].P \vdash \Gamma, x : ?_1 A}\ ?_1 \qquad \frac{P \vdash \Gamma, y : A}{!x(y).P \vdash \Gamma, x : !_1 A}\ !_1$$

$$\frac{P \vdash \Gamma, x : ?_m A, y : ?_n A}{P\{x/y\} \vdash \Gamma, x : ?_{m+n} A}\ \textsc{Contract}$$

$$\frac{P \vdash \Gamma, x : !_m A \qquad Q \vdash \Delta, x : !_n A}{(P \mid Q) \vdash \Gamma, \Delta, x : !_{m+n} A}\ \textsc{Pool}$$