Type Theory and NLP

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An abstract NLU-pipeline

$$\boxed{\text{Morphological}} \downarrow \boxed{\text{Lexical}} \downarrow \boxed{\text{Syntactic}} \downarrow \boxed{\text{Semantic}} \downarrow \boxed{\text{Pragmatic}}$$

"Mary saw foxes."

$\downarrow$

Mary see.PAST fox.PL

$\downarrow$

Mary:NP see:TV.PAST fox:NP.PL

$\downarrow$

Mary:NP [see:TV.PAST fox:NP.PL]

$\downarrow$

$$\exists p.\forall x.p(x) \supset (\mathbf{fox}(x) \wedge \mathbf{past}(\mathbf{see}(\mathrm{mary}, x)))$$

$\downarrow$

$\ldots$

A simple semantic calculus

$$\begin{aligned}
\text{Type } A, B &\coloneqq \mathbf{e} \mid \mathbf{t} \mid A \rightarrow B \\
\text{Term } M, N &\coloneqq x \mid C \mid \lambda x.M \mid (M\ N) \\
\text{Constant } C &\coloneqq \forall \mid \exists \mid \neg \mid \supset \mid \wedge \mid \vee \mid \ \ldots
\end{aligned}$$

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A}\ \text{Ax}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \rightarrow B} \rightarrow\text{I} \qquad \frac{\Gamma \vdash M : A \rightarrow B \qquad \Gamma \vdash N : A}{\Gamma \vdash (M\ N) : B} \rightarrow\text{E}$$

A simple semantic calculus

$$
\begin{aligned}
\text{Type } A, B &\coloneqq \mathbf{e} \mid \mathbf{t} \mid A \to B \\
\text{Term } M, N &\coloneqq x \mid C \mid \lambda x.M \mid (M\ N) \\
\text{Constant } C &\coloneqq \forall \mid \exists \mid \neg \mid \supset \mid \wedge \mid \vee \mid \ \ldots
\end{aligned}
$$

$$
\begin{aligned}
\supset &: \mathbf{ttt} \\
\forall &: (\mathbf{et})\mathbf{t} \\
\forall &: ((\mathbf{et})\mathbf{t})\mathbf{t} \\
\forall &: (\alpha\mathbf{t})\mathbf{t}
\end{aligned}
\qquad
\begin{aligned}
\forall x.M &\coloneqq \forall(\lambda x.M) \\
\exists x.M &\coloneqq \exists(\lambda x.M) \\
&\vdots
\end{aligned}
$$

An example

$$\dfrac{\dfrac{\dfrac{\text{saw}}{\mathbf{e},\mathbf{eet},\mathbf{e}\vdash\mathbf{eet}}\,\text{Ax}\qquad\dfrac{\text{foxes}}{\mathbf{e},\mathbf{eet},\mathbf{e}\vdash\mathbf{e}}\,\text{Ax}}{\mathbf{e},\mathbf{eet},\mathbf{e}\vdash\mathbf{et}}\,{\to}\text{E}\qquad\dfrac{\text{mary}}{\mathbf{e},\mathbf{eet},\mathbf{e}\vdash\mathbf{e}}\,\text{Ax}}{\mathbf{e},\mathbf{eet},\mathbf{e}\vdash\mathbf{t}}\,{\to}\text{E}$$

$$\Downarrow$$

$$\text{mary} : \mathbf{e}, \text{saw} : \mathbf{eet}, \text{foxes} : \mathbf{e} \vdash ((\text{saw foxes})\ \text{mary}) : \mathbf{t}$$

An example

$$\dfrac{\dfrac{\dfrac{\text{saw}}{\mathbf{e}, \mathbf{eet}, \mathbf{e} \vdash \mathbf{eet}}\,\text{Ax} \qquad \dfrac{\text{mary}}{\mathbf{e}, \mathbf{eet}, \mathbf{e} \vdash \mathbf{e}}\,\text{Ax}}{\mathbf{e}, \mathbf{eet}, \mathbf{e} \vdash \mathbf{et}}\,{\to}\text{E} \qquad \dfrac{\text{foxes}}{\mathbf{e}, \mathbf{eet}, \mathbf{e} \vdash \mathbf{e}}\,\text{Ax}}{\mathbf{e}, \mathbf{eet}, \mathbf{e} \vdash \mathbf{t}}\,{\to}\text{E}$$

$$\Downarrow$$

$$\text{mary} : \mathbf{e}, \text{saw} : \mathbf{eet}, \text{foxes} : \mathbf{e} \vdash ((\text{saw foxes})\ \text{mary}) : \mathbf{t}$$
$$\text{mary} : \mathbf{e}, \text{saw} : \mathbf{eet}, \text{foxes} : \mathbf{e} \vdash ((\text{saw mary})\ \text{foxes}) : \mathbf{t}$$

An example

$$\dfrac{\dfrac{\dfrac{\text{saw}}{\mathbf{e},\mathbf{eet},\mathbf{e}\vdash\mathbf{eet}}\,\text{Ax}\qquad \dfrac{\text{mary}}{\mathbf{e},\mathbf{eet},\mathbf{e}\vdash\mathbf{e}}\,\text{Ax}}{\mathbf{e},\mathbf{eet},\mathbf{e}\vdash\mathbf{et}}\,{\to}\text{E}\qquad \dfrac{\text{mary}}{\mathbf{e},\mathbf{eet},\mathbf{e}\vdash\mathbf{e}}\,\text{Ax}}{\mathbf{e},\mathbf{eet},\mathbf{e}\vdash\mathbf{t}}\,{\to}\text{E}$$

$$\Downarrow$$

$$\text{mary} : \mathbf{e}, \text{saw} : \mathbf{eet}, \text{foxes} : \mathbf{e} \vdash ((\text{saw foxes})\ \text{mary}) : \mathbf{t}$$
$$\text{mary} : \mathbf{e}, \text{saw} : \mathbf{eet}, \text{foxes} : \mathbf{e} \vdash ((\text{saw mary})\ \text{foxes}) : \mathbf{t}$$
$$\text{mary} : \mathbf{e}, \text{saw} : \mathbf{eet}, \text{foxes} : \mathbf{e} \vdash ((\text{saw mary})\ \text{mary}) : \mathbf{t}$$

An example

$$\dfrac{\dfrac{\dfrac{\text{saw}}{\mathbf{e},\mathbf{eet},\mathbf{e}\vdash\mathbf{eet}}\,\text{Ax}\qquad\dfrac{\text{foxes}}{\mathbf{e},\mathbf{eet},\mathbf{e}\vdash\mathbf{e}}\,\text{Ax}}{\mathbf{e},\mathbf{eet},\mathbf{e}\vdash\mathbf{et}}\,{\to}\text{E}\qquad\dfrac{\text{foxes}}{\mathbf{e},\mathbf{eet},\mathbf{e}\vdash\mathbf{e}}\,\text{Ax}}{\mathbf{e},\mathbf{eet},\mathbf{e}\vdash\mathbf{t}}\,{\to}\text{E}$$

$$\Downarrow$$

$$\begin{array}{l}
\text{mary} : \mathbf{e}, \text{saw} : \mathbf{eet}, \text{foxes} : \mathbf{e} \vdash ((\text{saw foxes})\ \text{mary}) : \mathbf{t}\\
\text{mary} : \mathbf{e}, \text{saw} : \mathbf{eet}, \text{foxes} : \mathbf{e} \vdash ((\text{saw mary})\ \text{foxes}) : \mathbf{t}\\
\text{mary} : \mathbf{e}, \text{saw} : \mathbf{eet}, \text{foxes} : \mathbf{e} \vdash ((\text{saw mary})\ \text{mary}) : \mathbf{t}\\
\text{mary} : \mathbf{e}, \text{saw} : \mathbf{eet}, \text{foxes} : \mathbf{e} \vdash ((\text{saw foxes})\ \text{foxes}) : \mathbf{t}
\end{array}$$

No implicit set structure

$$\text{Structure } \Gamma, \Delta, \Pi \coloneqq \varnothing \mid A \mid \Gamma \bullet \Delta$$

$$\frac{}{A \vdash A}\,\text{Ax}$$

$$\frac{\Gamma \bullet A \vdash B}{\Gamma \vdash A \to B}\,{\to}\text{I} \qquad \frac{\Gamma \vdash A \to B \qquad \Delta \vdash A}{\Gamma \bullet \Delta \vdash B}\,{\to}\text{E}$$

$$\frac{\Sigma[\Gamma \bullet \Gamma] \vdash B}{\Sigma[\Gamma] \vdash B}\,\text{Cont.} \qquad \frac{\Sigma[\Gamma] \vdash B}{\Sigma[\Gamma \bullet \Delta] \vdash B}\,\text{Weak.}$$

$$\frac{\Sigma[\Gamma \bullet \varnothing] \vdash B}{\Sigma[\Gamma] \vdash B}\,\varnothing\text{E}$$

$$\frac{\Sigma[\Delta \bullet \Gamma] \vdash B}{\Sigma[\Gamma \bullet \Delta] \vdash B}\,\text{Comm.} \qquad \frac{\Sigma[(\Gamma \bullet \Delta) \bullet \Pi] \vdash B}{\overline{\Sigma[\Gamma \bullet (\Delta \bullet \Pi)] \vdash B}}\,\text{Ass.}$$

No implicit set structure

$$\text{Structure } \Gamma, \Delta, \Pi \coloneqq \varnothing \mid A \mid \Gamma \bullet \Delta$$

$$\frac{}{A \vdash A}\,\text{Ax}$$

$$\frac{\Gamma \bullet A \vdash B}{\Gamma \vdash A \to B}\,{\to}\text{I} \qquad \frac{\Gamma \vdash A \to B \qquad \Delta \vdash A}{\Gamma \bullet \Delta \vdash B}\,{\to}\text{E}$$

$$\frac{\Sigma[\Gamma \bullet \Gamma] \vdash B}{\Sigma[\Gamma] \vdash B}\,\text{Cont.} \qquad \frac{\Sigma[\Gamma] \vdash B}{\Sigma[\Gamma \bullet \Delta] \vdash B}\,\text{Weak.}$$

$$\frac{\Sigma[\Gamma \bullet \varnothing] \vdash B}{\Sigma[\Gamma] \vdash B}\,\varnothing\text{E}$$

$$\frac{\Sigma[\Delta \bullet \Gamma] \vdash B}{\Sigma[\Gamma \bullet \Delta] \vdash B}\,\text{Comm.} \qquad \frac{\Sigma[(\Gamma \bullet \Delta) \bullet \Pi] \vdash B}{\Sigma[\Gamma \bullet (\Delta \bullet \Pi)] \vdash B}\,\text{Ass.}$$

No implicit set structure

$$\text{Structure } \Gamma, \Delta, \Pi \coloneqq \varnothing \mid A \mid \Gamma \bullet \Delta$$

$$\frac{}{A \vdash A}\text{Ax}$$

$$\frac{\Gamma \bullet A \vdash B}{\Gamma \vdash A \to B}\to\text{I} \qquad \frac{\Gamma \vdash A \to B \qquad \Delta \vdash A}{\Gamma \bullet \Delta \vdash B}\to\text{E}$$

$$\frac{\Sigma[\Gamma \bullet \Gamma] \vdash B}{\Sigma[\Gamma] \vdash B}\text{Cont.} \qquad \frac{\Sigma[\Gamma] \vdash B}{\Sigma[\Gamma \bullet \Delta] \vdash B}\text{Weak.}$$

$$\frac{\Sigma[\Gamma \bullet \varnothing] \vdash B}{\Sigma[\Gamma] \vdash B}\varnothing\text{E}$$

$$\frac{\Sigma[\Delta \bullet \Gamma] \vdash B}{\Sigma[\Gamma \bullet \Delta] \vdash B}\text{Comm.} \qquad \frac{\Sigma[(\Gamma \bullet \Delta) \bullet \Pi] \vdash B}{\Sigma[\Gamma \bullet (\Delta \bullet \Pi)] \vdash B}\text{Ass.}$$

An example

$$\dfrac{\dfrac{\dfrac{\dfrac{\text{saw}}{\mathbf{eet} \vdash \mathbf{eet}}\,\text{Ax} \qquad \dfrac{\text{foxes}}{\mathbf{e} \vdash \mathbf{e}}\,\text{Ax}}{\mathbf{eet} \bullet \mathbf{e} \vdash \mathbf{et}}\,{\to}\text{E} \qquad \dfrac{\text{mary}}{\mathbf{e} \vdash \mathbf{e}}\,\text{Ax}}{(\mathbf{eet} \bullet \mathbf{e}) \bullet \mathbf{e} \vdash \mathbf{et}}\,{\to}\text{E}}{\mathbf{e} \bullet (\mathbf{eet} \bullet \mathbf{e}) \vdash \mathbf{et}}\,\text{Comm.}$$

$$\Downarrow$$

$$\text{mary} : \mathbf{e}, \text{saw} : \mathbf{eet}, \text{foxes} : \mathbf{e} \vdash ((\text{saw foxes})\ \text{mary}) : \mathbf{t}$$

An example

$$
\dfrac{
  \dfrac{
    \dfrac{\text{saw}}{\mathbf{eet} \vdash \mathbf{eet}}\,\text{Ax}
    \quad
    \dfrac{\text{mary}}{\color{red}{\mathbf{e}} \vdash \mathbf{e}}\,\text{Ax}
  }{\mathbf{eet} \bullet \color{red}{\mathbf{e}} \vdash \mathbf{et}}\,{\to}\text{E}
  \quad
  \dfrac{\text{foxes}}{\color{green}{\mathbf{e}} \vdash \mathbf{e}}\,\text{Ax}
}{
  \dfrac{
    \dfrac{(\mathbf{eet} \bullet \color{red}{\mathbf{e}}) \bullet \color{green}{\mathbf{e}} \vdash \mathbf{et}}{(\color{red}{\mathbf{e}} \bullet \mathbf{eet}) \bullet \color{green}{\mathbf{e}} \vdash \mathbf{et}}\,\text{Comm.}
  }{\color{red}{\mathbf{e}} \bullet (\mathbf{eet} \bullet \color{green}{\mathbf{e}}) \vdash \mathbf{et}}\,\text{Ass.}
}\,{\to}\text{E}
$$

$$\Downarrow$$

$$\text{mary} : \mathbf{e}, \text{saw} : \mathbf{eet}, \text{foxes} : \mathbf{e} \vdash ((\text{saw mary})\ \text{foxes}) : \mathbf{t}$$

An example

$$
\begin{prooftree}
\AxiomC{saw}
\RightLabel{Ax}
\UnaryInfC{$\mathbf{eet} \vdash \mathbf{eet}$}
\AxiomC{mary}
\RightLabel{Ax}
\UnaryInfC{$\color{red}{\mathbf{e}}\color{black} \vdash \mathbf{e}$}
\RightLabel{$\to$E}
\BinaryInfC{$\mathbf{eet} \bullet \color{red}{\mathbf{e}}\color{black} \vdash \mathbf{et}$}
\AxiomC{mary}
\RightLabel{Ax}
\UnaryInfC{$\color{red}{\mathbf{e}}\color{black} \vdash \mathbf{e}$}
\RightLabel{$\to$E}
\BinaryInfC{$(\mathbf{eet} \bullet \color{red}{\mathbf{e}}\color{black}) \bullet \color{red}{\mathbf{e}}\color{black} \vdash \mathbf{et}$}
\RightLabel{Ass.}
\UnaryInfC{$\mathbf{eet} \bullet (\color{red}{\mathbf{e}}\color{black} \bullet \color{red}{\mathbf{e}}\color{black}) \vdash \mathbf{et}$}
\RightLabel{Comm.}
\UnaryInfC{$(\color{red}{\mathbf{e}}\color{black} \bullet \color{red}{\mathbf{e}}\color{black}) \bullet \mathbf{eet} \vdash \mathbf{et}$}
\RightLabel{Cont.}
\UnaryInfC{$\color{red}{\mathbf{e}}\color{black} \bullet \mathbf{eet} \bullet \vdash \mathbf{et}$}
\RightLabel{Weak.}
\UnaryInfC{$\color{red}{\mathbf{e}}\color{black} \bullet (\mathbf{eet} \bullet \color{green}{\mathbf{e}}\color{black}) \vdash \mathbf{et}$}
\end{prooftree}
$$

$$\Downarrow$$

$$\text{mary} : \mathbf{e}, \text{saw} : \mathbf{eet}, \text{foxes} : \mathbf{e} \vdash ((\text{saw mary})\ \text{mary}) : \mathbf{t}$$

An example

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\text{saw}}{\mathbf{eet} \vdash \mathbf{eet}}\,\text{Ax} \qquad \cfrac{\text{foxes}}{\mathbf{e} \vdash \mathbf{e}}\,\text{Ax}}{\mathbf{eet} \bullet \mathbf{e} \vdash \mathbf{et}}\,{\to}\text{E} \qquad \cfrac{\text{foxes}}{\mathbf{e} \vdash \mathbf{e}}\,\text{Ax}}{(\mathbf{eet} \bullet \mathbf{e}) \bullet \mathbf{e} \vdash \mathbf{et}}\,{\to}\text{E}}{\mathbf{eet} \bullet (\mathbf{e} \bullet \mathbf{e}) \vdash \mathbf{et}}\,\text{Ass.}}{\mathbf{eet} \bullet \mathbf{e} \vdash \mathbf{et}}\,\text{Cont.}}{(\mathbf{eet} \bullet \mathbf{e}) \bullet \mathbf{e} \vdash \mathbf{et}}\,\text{Weak.}}{\mathbf{e} \bullet (\mathbf{eet} \bullet \mathbf{e}) \vdash \mathbf{et}}\,\text{Comm.}
$$

$$\Downarrow$$

$$\text{mary} : \mathbf{e}, \text{saw} : \mathbf{eet}, \text{foxes} : \mathbf{e} \vdash ((\text{saw foxes})\ \text{foxes}) : \mathbf{t}$$

A simple syntactic calculus

$$\text{Type } A, B \coloneqq \mathsf{S} \mid \mathsf{N} \mid \mathsf{NP} \mid A \backslash B \mid B \,/\, A$$
$$\text{Structure } \Gamma, \Delta \coloneqq A \mid \Gamma \bullet \Delta$$

$$\frac{}{A \vdash A}\,\mathsf{Ax}$$

$$\frac{A \bullet \Gamma \vdash B}{\Gamma \vdash A \backslash B}\,\backslash\mathsf{I} \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash A \backslash B}{\Gamma \bullet \Delta \vdash B}\,\backslash\mathsf{E}$$

$$\frac{\Gamma \bullet A \vdash B}{\Gamma \vdash B \,/\, A}\,/\mathsf{I} \qquad \frac{\Gamma \vdash B \,/\, A \qquad \Delta \vdash A}{\Gamma \bullet \Delta \vdash B}\,/\mathsf{E}$$

An example

$$
\dfrac{\dfrac{\text{mary}}{NP \vdash NP}\,Ax \qquad \dfrac{\dfrac{\text{saw}}{(NP \setminus S) / NP \vdash (NP \setminus S) / NP}\,Ax \qquad \dfrac{\text{foxes}}{NP \vdash NP}\,Ax}{(NP \setminus S) / NP \bullet NP \vdash NP \setminus S}\,/E}{NP \bullet ((NP \setminus S) / NP \bullet NP) \vdash S}\,\setminus E
$$

$$\Downarrow$$

?

From syntactic to semantic calculus

$$
\boxed{
\begin{array}{llll}
\mathsf{S}^* & \mapsto \mathbf{t} & & \\
\mathsf{N}^* & \mapsto \mathbf{e} \to \mathbf{t} & \mathsf{NP}^* & \mapsto \mathbf{e} \\
(A \setminus B)^* & \mapsto A^* \to B^* & (B \mathbin{/} A)^* & \mapsto A^* \to B^*
\end{array}
}
$$

$$
\frac{}{A \vdash A}\,\mathsf{Ax} \quad \Longrightarrow \quad \frac{}{A^* \vdash A^*}\,\mathsf{Ax}
$$

From syntactic to semantic calculus

S^*	$\mapsto \mathbf{t}$		
N^*	$\mapsto \mathbf{e} \to \mathbf{t}$	NP^*	$\mapsto \mathbf{e}$
$(A \setminus B)^*$	$\mapsto A^* \to B^*$	$(B \,/\, A)^*$	$\mapsto A^* \to B^*$

$$\frac{A \bullet \Gamma \vdash B}{\Gamma \vdash A \setminus B}\setminus\mathrm{I} \quad\Longrightarrow\quad \dfrac{\dfrac{A^* \bullet \Gamma^* \vdash B^*}{\Gamma^* \bullet A^* \vdash B^*}\,\text{Comm.}}{\Gamma^* \vdash A^* \to B^*}\to\mathrm{I}$$

$$\frac{\Gamma \bullet A \vdash B}{\Gamma \vdash B \,/\, A}/\mathrm{I} \quad\Longrightarrow\quad \frac{\Gamma^* \bullet A^* \vdash B^*}{\Gamma^* \vdash A^* \to B^*}\to\mathrm{I}$$

From syntactic to semantic calculus

S^*	$\mapsto \mathbf{t}$		
N^*	$\mapsto \mathbf{e} \to \mathbf{t}$	NP^*	$\mapsto \mathbf{e}$
$(A \setminus B)^*$	$\mapsto A^* \to B^*$	$(B \,/\, A)^*$	$\mapsto A^* \to B^*$

$$\frac{\Gamma \vdash A \qquad \Delta \vdash A \setminus B}{\Gamma \bullet \Delta \vdash B} \setminus\mathsf{E} \quad \Longrightarrow \quad \frac{\dfrac{\Delta^* \vdash A^* \to B^* \qquad \Gamma^* \vdash A^*}{\Delta^* \bullet \Gamma^* \vdash B^*} \to\mathsf{E}}{\Gamma^* \bullet \Delta^* \vdash B^*} \text{Comm.}$$

$$\frac{\Gamma \vdash B \,/\, A \qquad \Delta \vdash A}{\Gamma \bullet \Delta \vdash B} /\mathsf{E} \quad \Longrightarrow \quad \frac{\Gamma^* \vdash A^* \to B^* \qquad \Delta^* \vdash A^*}{\Gamma^* \bullet \Delta^* \vdash B^*} \to\mathsf{E}$$

An example

$$
\dfrac{\dfrac{\text{mary}}{NP \vdash NP}\,Ax \qquad \dfrac{\dfrac{\text{saw}}{(NP \setminus S) / NP \vdash (NP \setminus S) / NP}\,Ax \qquad \dfrac{\text{foxes}}{NP \vdash NP}\,Ax}{(NP \setminus S) / NP \bullet NP \vdash NP \setminus S}\,/E}{NP \bullet ((NP \setminus S) / NP \bullet NP) \vdash S}\,\setminus E
$$

$$\Downarrow$$

$$\text{mary} : \mathbf{e}, \text{saw} : \mathbf{eet}, \text{foxes} : \mathbf{e} \vdash ((\text{saw foxes})\ \text{mary}) : \mathbf{t}$$

Joachim Lambek (1922–2014)

Display calculus

Generalises the sequent calculus;

Generic proof of cut-elimination;

$$\frac{\Gamma \vdash A \qquad A \vdash \Delta}{\Gamma \vdash \Delta}\,\mathrm{Cut}$$

Decidable, easy-to-implement proof search;

Focusing can be used to restrict spurious ambiguity.

Display calculus

$$
\begin{array}{ll}
\text{Structure}^{+}\ \Gamma & := \cdot A \cdot \mid \Gamma_1 \bullet \Gamma_2 \\
\text{Structure}^{-}\ \Delta & := \cdot A \cdot \mid \Gamma \setminus \Delta \mid \Delta \mathbin{/} \Gamma
\end{array}
$$

$$\frac{}{\cdot A \cdot \vdash \cdot A \cdot}\,\mathsf{Ax}$$

$$\frac{\Gamma \vdash \cdot A \cdot \qquad \cdot B \cdot \vdash \Delta}{\cdot A \setminus B \cdot \vdash \Gamma \setminus \Delta}\,\mathsf{L}\backslash \qquad \frac{\Gamma \vdash \cdot A \cdot \setminus \cdot B \cdot}{\Gamma \vdash \cdot A \setminus B \cdot}\,\mathsf{R}\backslash$$

$$\frac{\Gamma \vdash \cdot A \cdot \qquad \cdot B \cdot \vdash \Delta}{\cdot B \mathbin{/} A \cdot \vdash \Delta \mathbin{/} \Gamma}\,\mathsf{L}/ \qquad \frac{\Gamma \vdash \cdot B \cdot \mathbin{/} \cdot A \cdot}{\Gamma \vdash \cdot B \mathbin{/} A \cdot}\,\mathsf{R}/$$

$$\frac{\Gamma_2 \vdash \Gamma_1 \setminus \Delta}{\Gamma_1 \bullet \Gamma_2 \vdash \Delta}\,\mathsf{Res}\backslash\bullet \qquad \frac{\Gamma_1 \vdash \Delta \mathbin{/} \Gamma_2}{\Gamma_1 \bullet \Gamma_2 \vdash \Delta}\,\mathsf{Res}/\bullet$$

Display calculus

$$
\begin{aligned}
\text{Structure}^{+}\ \Gamma \quad &:= \cdot A \cdot \mid \Gamma_1 \bullet \Gamma_2 \\
\text{Structure}^{-}\ \Delta \quad &:= \cdot A \cdot \mid \Gamma \mathbin{\backslash} \Delta \mid \Delta \mathbin{/} \Gamma
\end{aligned}
$$

$$\frac{}{\cdot A \cdot \vdash \cdot A \cdot}\,\text{Ax}$$

$$\frac{\Gamma \vdash \cdot A \cdot \qquad \cdot B \cdot \vdash \Delta}{\cdot A \mathbin{\backslash} B \cdot \vdash \Gamma \mathbin{\backslash} \Delta}\,\text{L}\backslash \qquad \frac{\Gamma \vdash \cdot A \cdot \mathbin{\backslash} \cdot B \cdot}{\Gamma \vdash \cdot A \mathbin{\backslash} B \cdot}\,\text{R}\backslash$$

$$\frac{\Gamma \vdash \cdot A \cdot \qquad \cdot B \cdot \vdash \Delta}{\cdot B \mathbin{/} A \cdot \vdash \Delta \mathbin{/} \Gamma}\,\text{L}/ \qquad \frac{\Gamma \vdash \cdot B \cdot \mathbin{/} \cdot A \cdot}{\Gamma \vdash \cdot B \mathbin{/} A \cdot}\,\text{R}/$$

$$\frac{\Gamma_2 \vdash \Gamma_1 \mathbin{\backslash} \Delta}{\Gamma_1 \bullet \Gamma_2 \vdash \Delta}\,\text{Res}\backslash\bullet \qquad \frac{\Gamma_1 \vdash \Delta \mathbin{/} \Gamma_2}{\Gamma_1 \bullet \Gamma_2 \vdash \Delta}\,\text{Res}/\bullet$$

Display calculus

$$
\begin{aligned}
\text{Structure}^{+}\ \Gamma &\quad := \cdot A \cdot \mid \Gamma_1 \bullet \Gamma_2 \\
\text{Structure}^{-}\ \Delta &\quad := \cdot A \cdot \mid \Gamma \setminus \Delta \mid \Delta \,/\, \Gamma
\end{aligned}
$$

$$\frac{}{\cdot A \cdot \vdash \cdot A \cdot}\,\text{Ax}$$

$$\frac{\Gamma \vdash \cdot A \cdot \qquad \cdot B \cdot \vdash \Delta}{\cdot A \setminus B \cdot \vdash \Gamma \setminus \Delta}\,\text{L}\setminus \qquad \frac{\Gamma \vdash \cdot A \cdot \setminus \cdot B \cdot}{\Gamma \vdash \cdot A \setminus B \cdot}\,\text{R}\setminus$$

$$\frac{\Gamma \vdash \cdot A \cdot \qquad \cdot B \cdot \vdash \Delta}{\cdot B \,/\, A \cdot \vdash \Delta \,/\, \Gamma}\,\text{L}/ \qquad \frac{\Gamma \vdash \cdot B \cdot \,/\, \cdot A \cdot}{\Gamma \vdash \cdot B \,/\, A \cdot}\,\text{R}/$$

$$\boxed{\frac{\Gamma_2 \vdash \Gamma_1 \setminus \Delta}{\Gamma_1 \bullet \Gamma_2 \vdash \Delta}\,\text{Res}\setminus\bullet \qquad \frac{\Gamma_1 \vdash \Delta \,/\, \Gamma_2}{\Gamma_1 \bullet \Gamma_2 \vdash \Delta}\,\text{Res}/\bullet}$$

An example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cdot \mathrm{NP}\cdot \vdash \cdot \mathrm{NP}\cdot}\,\mathrm{Ax} \qquad \dfrac{}{\cdot \mathrm{S}\cdot \vdash \cdot \mathrm{S}\cdot}\,\mathrm{Ax}}{\cdot \mathrm{NP} \backslash \mathrm{S}\cdot \vdash \cdot \mathrm{NP}\cdot \backslash \cdot \mathrm{S}\cdot}\,\mathrm{L}\backslash \qquad \dfrac{}{\cdot \mathrm{NP}\cdot \vdash \cdot \mathrm{NP}\cdot}\,\mathrm{Ax}}{\cdot(\mathrm{NP} \backslash \mathrm{S}) / \mathrm{NP}\cdot \vdash (\cdot \mathrm{NP}\cdot \backslash \cdot \mathrm{S}\cdot) / \cdot \mathrm{NP}\cdot}\,\mathrm{L}/}{\cdot(\mathrm{NP} \backslash \mathrm{S}) / \mathrm{NP}\cdot \bullet \cdot \mathrm{NP}\cdot \vdash \cdot \mathrm{NP}\cdot \backslash \cdot \mathrm{S}\cdot}\,\mathrm{Res}/\bullet}{\cdot \mathrm{NP}\cdot \bullet \cdot(\mathrm{NP} \backslash \mathrm{S}) / \mathrm{NP}\cdot \bullet \cdot \mathrm{NP}\cdot \vdash \cdot \mathrm{S}\cdot}\,\mathrm{Res}\backslash\bullet$$

$$\Downarrow$$

?

From display calculus to semantic calculus

$$\begin{aligned}
(\cdot A \cdot)^* &\mapsto A^* & \qquad (\cdot A \cdot)^{**} &\mapsto A^* \\
(\Gamma_1 \bullet \Gamma_2)^* &\mapsto \Gamma_1^* \bullet \Gamma_2^* & \qquad (\Gamma_1 \bullet \Gamma_2)^{**} &\mapsto \Gamma_1^{**} \times \Gamma_2^{**}
\end{aligned}$$

$$\begin{aligned}
(\cdot A \cdot)^* &\mapsto A^* & & \\
(\Delta \mathbin{/} \Gamma)^* &\mapsto \Gamma^{**} \to \Delta^* & & \\
(\Gamma \mathbin{\backslash} \Delta)^* &\mapsto \Gamma^{**} \to \Delta^* & \qquad (\Gamma \vdash \Delta)^* &\mapsto \Gamma^* \vdash \Delta^*
\end{aligned}$$

From display calculus to semantic calculus

$$\frac{\Gamma \vdash \cdot A \cdot \qquad \cdot B \cdot \vdash \Delta}{\cdot A \setminus B \cdot \vdash \Gamma \setminus \Delta}\,\mathrm{L}\setminus \quad , \quad \frac{\Gamma \vdash \cdot A \cdot \qquad \cdot B \cdot \vdash \Delta}{\cdot B \,/\, A \cdot \vdash \Delta \,/\, \Gamma}\,\mathrm{L}/$$

$$\Downarrow$$

$$\dfrac{\dfrac{\dfrac{B^* \vdash \Delta^*}{\varnothing \vdash B^* \to \Delta^*}\to\mathrm{I} \qquad \dfrac{\dfrac{}{A^* \to B^* \vdash A^* \to B^*}\,\mathrm{Ax} \qquad \Gamma^{**} \vdash A^*}{A^* \to B^* \bullet \Gamma^{**} \vdash B^*}\to\mathrm{E}}{A^* \to B^* \bullet \Gamma^{**} \vdash \Delta^*}\to\mathrm{E}}{A^* \to B^* \vdash \Gamma^{**} \to \Delta^*}\to\mathrm{I}$$

From display calculus to semantic calculus

$$\frac{\Gamma \vdash \cdot A \cdot \setminus \cdot B \cdot}{\Gamma \vdash \cdot A \setminus B \cdot}\,\mathsf{R}\setminus \quad , \quad \frac{\Gamma \vdash \cdot B \cdot \,/\, \cdot A \cdot}{\Gamma \vdash \cdot B \,/\, A \cdot}\,\mathsf{R}/$$

$$\Downarrow$$

$$\Gamma^{**} \vdash B^{*} \to A^{*}$$

From display calculus to semantic calculus

$$\frac{\Gamma_1 \vdash \Delta \mathbin{/} \Gamma_2}{\Gamma_1 \bullet \Gamma_2 \vdash \Delta}\,\mathrm{Res}/\bullet \qquad\qquad \frac{\Gamma_1 \bullet \Gamma_2 \vdash \Delta}{\Gamma_1 \vdash \Delta \mathbin{/} \Gamma_2}\,\mathrm{Res}\bullet/$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\frac{\Gamma_1^{**} \vdash \Gamma_2^{**} \to \Delta^* \qquad \overline{\Gamma_2^{**} \vdash \Gamma_2^{**}}\,\mathrm{Ax}}{\Gamma_1^{**} \bullet \Gamma_2^{**} \vdash \Delta^*}\,{\to}\mathrm{E} \qquad \frac{\Gamma_1^{**} \bullet \Gamma_2^{**} \vdash \Delta^*}{\Gamma_1^{**} \vdash \Gamma_2^{**} \to \Delta^*}\,{\to}\mathrm{I}$$

From display calculus to semantic calculus

$$\frac{\Gamma_2 \vdash \Gamma_1 \setminus \Delta}{\Gamma_1 \bullet \Gamma_2 \vdash \Delta}\,\text{Res}\backslash\bullet \qquad\qquad \frac{\Gamma_1 \bullet \Gamma_2 \vdash \Delta}{\Gamma_2 \vdash \Gamma_1 \setminus \Delta}\,\text{Res}\bullet\backslash$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\frac{\dfrac{\Gamma_2^{**} \vdash \Gamma_1^{**} \to \Delta^* \qquad \dfrac{}{\Gamma_1^{**} \vdash \Gamma_1^{**}}\,\text{Ax}}{\Gamma_2^{**} \bullet \Gamma_1^{**} \vdash \Delta^*}\,{\to}\text{E}}{\Gamma_1^{**} \bullet \Gamma_2^{**} \vdash \Delta^*}\,\text{Comm.} \qquad \frac{\dfrac{\Gamma_1^{**} \bullet \Gamma_2^{**} \vdash \Delta^*}{\Gamma_2^{**} \bullet \Gamma_1^{**} \vdash \Delta^*}\,\text{Comm.}}{\Gamma_2^{**} \vdash \Gamma_1^{**} \to \Delta^*}\,{\to}\text{I}$$

An example

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\mathbf{t} \vdash \mathbf{t}$}
\RightLabel{$\to$I}
\UnaryInfC{$\varnothing \vdash \mathbf{tt}$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\mathbf{et} \vdash \mathbf{et}$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\mathbf{e} \vdash \mathbf{e}$}
\RightLabel{$\to$E}
\BinaryInfC{$\mathbf{et} \bullet \mathbf{e} \vdash \mathbf{t}$}
\RightLabel{$\to$E}
\BinaryInfC{$\mathbf{et} \bullet \mathbf{e} \vdash \mathbf{t}$}
\RightLabel{$\to$I}
\UnaryInfC{$\mathbf{et} \vdash \mathbf{et}$}
\RightLabel{$\to$I}
\UnaryInfC{$\varnothing \vdash (\mathbf{et})\mathbf{et}$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\mathbf{eet} \vdash \mathbf{eet}$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\mathbf{e} \vdash \mathbf{e}$}
\RightLabel{$\to$E}
\BinaryInfC{$\mathbf{eet} \bullet \mathbf{e} \vdash \mathbf{et}$}
\RightLabel{$\to$E}
\BinaryInfC{$\mathbf{eet} \bullet \mathbf{e} \vdash \mathbf{et}$}
\RightLabel{$\to$I}
\UnaryInfC{$\mathbf{eet} \vdash \mathbf{eet}$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\mathbf{e} \vdash \mathbf{e}$}
\RightLabel{$\to$E}
\BinaryInfC{$\mathbf{eet} \bullet \mathbf{e} \vdash \mathbf{et}$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\mathbf{e} \vdash \mathbf{e}$}
\RightLabel{$\to$E}
\BinaryInfC{$(\mathbf{eet} \bullet \mathbf{e}) \bullet \mathbf{e} \vdash \mathbf{t}$}
\RightLabel{Comm.}
\UnaryInfC{$\mathbf{e} \bullet (\mathbf{eet} \bullet \mathbf{e}) \vdash \mathbf{t}$}
\end{prooftree}
$$

$$\Downarrow$$

$$(((\lambda x.(\lambda k.\lambda y.(\lambda z.z)\ (k\ y))\ (\text{saw}\ x))\ \text{foxes})\ \text{mary})$$

An example

$$
\dfrac{\dfrac{\dfrac{}{\mathbf{eet} \vdash \mathbf{eet}}\,\text{Ax} \qquad \dfrac{}{\mathbf{e} \vdash \mathbf{e}}\,\text{Ax}}{\mathbf{eet} \bullet \mathbf{e} \vdash \mathbf{et}}\,{\to}\text{E} \qquad \dfrac{}{\mathbf{e} \vdash \mathbf{e}}\,\text{Ax}}{\dfrac{(\mathbf{eet} \bullet \mathbf{e}) \bullet \mathbf{e} \vdash \mathbf{t}}{\mathbf{e} \bullet (\mathbf{eet} \bullet \mathbf{e}) \vdash \mathbf{t}}\,\text{Comm.}}\,{\to}\text{E}
$$

$$\Downarrow$$

((saw foxes) mary)

Let's take a step back

We now have:

- a natural deduction semantic calculus;
- a display logic syntactic calculus;
- a decidable algorithm for proof search in the syntactic calculus;
- a translation from the syntactic to the semantic calculus.

If we put all these items together, we can build our semantic function!

$$\begin{gathered}\text{Mary:NP [see:TV.PAST fox:NP.PL]}\\ \downarrow\\ \boxed{\text{Semantic}}\\ \downarrow\\ \exists p.\forall x.p(x) \supset (\mathbf{fox}(x) \wedge \mathbf{past}(\mathbf{see}(\text{mary}, x)))\end{gathered}$$

Sometimes language doesn't *look* compositional

"I walked the dog."

'I' seems to refer to an entity that is always available, but whose denotation varies with context.

Sometimes language doesn't *look* compositional

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"I walked the damned dog."

'Damned' doesn't seem to interact directly with the sentence meaning, but conveys an additional meaning of dislike towards the dog or general annoyance.

Sometimes language doesn't *look* compositional

"<u>I</u> walked the dog."

'I' seems to refer to an entity that is always available, but whose denotation varies with context.

"I walked the <u>damned</u> dog."

'Damned' doesn't seem to interact directly with the sentence meaning, but conveys an additional meaning of dislike towards the dog or general annoyance.

"John left. <u>He</u> was whistling."

'He' seems to be able to refer to 'John' when the sentences in are in this order, but not when they're the other way around.

But we know better. . .

"I walked the dog."

'I' seems to refer to an entity that is always available, but whose denotation varies with context.

✓ **Reader Monad**

"I walked the damned dog."

'Damned' doesn't seem to interact directly with the sentence meaning, but conveys an additional meaning of dislike towards the dog or general annoyance.

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"I walked the damned dog."

'Damned' doesn't seem to interact directly with the sentence meaning, but conveys an additional meaning of dislike towards the dog or general annoyance.

✓ **Writer Monad**

"John left. He was whistling."

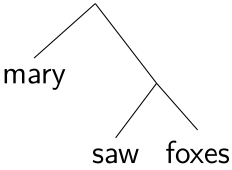
'He' seems to be able to refer to 'John' when the sentences in are in this order, but not when they're the other way around.

✓ **State Monad**

Quantifier Raising

"Mary saw <u>foxes</u>."

Given that the parse tree is:



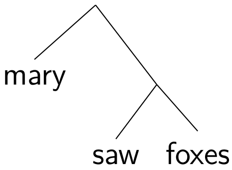
And the denotation is:

((see foxes) mary)

Quantifier Raising

"Mary saw foxes."

Given that the parse tree is:



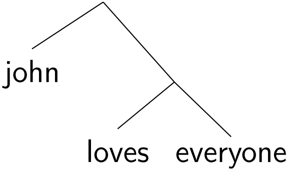
And the denotation is:

$$\exists p.\forall x.p(x) \supset (\mathbf{fox}(x) \wedge \mathbf{past}(\mathbf{see}(\mathrm{mary}, x)))$$

Quantifier Raising

"John loves <u>everyone</u>."

Given that the parse tree is:



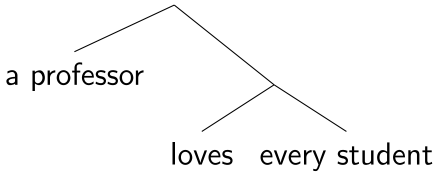
And the denotation is:

$$\forall x.\mathbf{person}(x) \supset \mathbf{love}(\text{john}, x)$$

Scope Ambiguity

"A professor talked to every student."

Given that the parse tree is:



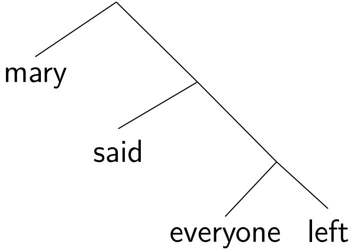
And the denotation is:

$$\exists x.\mathbf{professor}(x) \land (\forall y.\mathbf{student}(y) \supset \mathbf{talk}(x, y))$$
$$\forall y.\mathbf{student}(y) \supset (\exists x.\mathbf{professor}(x) \land \mathbf{talk}(x, y))$$

Scope Islands

"Mary said everyone left."

Given that the parse tree is:



And the denotation is:

$$\textbf{said}(\text{mary}, \forall x.\textbf{left}(x))$$

And definitely isn't:

$$\forall x.\textbf{said}(\text{mary}, \textbf{left}(x))$$

What could we do *right now*?

Use higher order functions, but:

- many different types

$$S \,/\, (NP \setminus S),\ ((NP \setminus S) \,/\, NP) \setminus (NP \setminus S),\ \ldots$$

Use a continuation monad, but:

- only *one* interpretation, so no scope ambiguity
- can only take scope at the top-level
- can not be delimited

Use delimited continuations, but:

- again, only *one* interpretation
- is not a monad, but an indexed monad, which has *three arguments*, so should be reflected in the syntactic calculus

Quantifier Raising

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{}{\cdot NP\cdot \vdash \cdot NP\cdot}\,\text{Ax}
          \quad
          \dfrac{}{\cdot S\cdot \vdash \cdot S\cdot}\,\text{Ax}
        }{\cdot NP \backslash S\cdot \vdash \cdot NP\cdot \backslash \cdot S\cdot}\,\text{L}\backslash
      }{\cdot NP\cdot \bullet \cdot NP \backslash S\cdot \vdash \cdot S\cdot}\,\text{Res}\backslash\bullet
    }{\cdot NP \backslash S\cdot \vdash \cdot NP\cdot \backslash \cdot S\cdot}\,\text{Res}\bullet\backslash
  }{\cdot NP \backslash S\cdot \vdash \cdot NP \backslash S\cdot}\,\text{R}\backslash
  \quad
  \dfrac{}{\cdot S\cdot \vdash \cdot S\cdot}\,\text{Ax}
}{
  \dfrac{\cdot S / (NP \backslash S)\cdot \vdash \cdot S\cdot / \cdot NP \backslash S\cdot}{\cdot S / (NP \backslash S)\cdot \bullet \cdot NP \backslash S\cdot \vdash \cdot S\cdot}\,\text{Res}/\bullet
}
$$

Quantifier Raising

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\cdot NP\cdot \vdash \cdot NP\cdot}\,\text{Ax} \quad \dfrac{}{\cdot S\cdot \vdash \cdot S\cdot}\,\text{Ax}}{\cdot NP \backslash S\cdot \vdash \cdot NP\cdot \backslash \cdot S\cdot}\,\text{L}\backslash}{\boxed{\cdot NP\cdot \bullet \cdot NP \backslash S\cdot \vdash \cdot S\cdot}}\,\text{Res}\backslash\bullet}{\cdot NP \backslash S\cdot \vdash \cdot NP\cdot \backslash \cdot S\cdot}\,\text{Res}\bullet\backslash}{\cdot NP \backslash S\cdot \vdash \cdot NP \backslash S\cdot}\,\text{R}\backslash \quad \dfrac{}{\cdot S\cdot \vdash \cdot S\cdot}\,\text{Ax}}{\cdot S / (NP \backslash S)\cdot \vdash \cdot S\cdot / \cdot NP \backslash S\cdot}}{\cdot S / (NP \backslash S)\cdot \bullet \cdot NP \backslash S\cdot \vdash \cdot S\cdot}\,\text{Res}/\bullet
$$

Quantifier Raising

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\cdot NP\cdot \bullet \cdot (NP \backslash S) / NP \cdot \bullet \cdot NP \cdot \vdash \cdot S \cdot$}
\RightLabel{$\downarrow$}
\UnaryInfC{$\cdot NP\cdot \bullet \cdot NP\cdot \bullet \cdot (NP \backslash S) / NP \cdot \bullet \mathbf{I} \vdash \cdot S \cdot$}
\RightLabel{Res$\bullet\backslash$}
\UnaryInfC{$\cdot NP\cdot \bullet \cdot (NP \backslash S) / NP \cdot \bullet \mathbf{I} \vdash \cdot NP \cdot \backslash \cdot S \cdot$}
\RightLabel{R$\backslash$}
\UnaryInfC{$\cdot NP\cdot \bullet \cdot (NP \backslash S) / NP \cdot \bullet \mathbf{I} \vdash \cdot NP \backslash S \cdot$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\cdot S \cdot \vdash \cdot S \cdot$}
\RightLabel{L$/$}
\BinaryInfC{$\cdot S / (NP \backslash S) \cdot \vdash \cdot S \cdot / (\cdot NP\cdot \bullet \cdot (NP \backslash S) / NP \cdot \bullet \mathbf{I})$}
\RightLabel{Res$/\bullet$}
\UnaryInfC{$\cdot S / (NP \backslash S) \cdot \bullet \cdot NP\cdot \bullet \cdot (NP \backslash S) / NP \cdot \bullet \mathbf{I} \vdash \cdot S \cdot$}
\RightLabel{$\uparrow$}
\UnaryInfC{$\cdot NP\cdot \bullet \cdot (NP \backslash S) / NP \cdot \bullet \cdot S / (NP \backslash S) \cdot \vdash \cdot S \cdot$}
\end{prooftree}
$$

Quantifier Raising

$$\frac{\Gamma \bullet \Sigma\,[\,\mathbf{I}\,] \vdash \Delta}{\Sigma\,[\,\Gamma\,] \vdash \Delta}\,\uparrow\downarrow$$

Quantifier Raising

$$
\begin{array}{cl}
\vdots & \\
\hline
\cdot NP \cdot \bullet \cdot (NP \backslash S) / NP \cdot \bullet \cdot NP \cdot \vdash \cdot S \cdot & \\
\hline
\cdot NP \cdot \bullet \cdot NP \cdot \bullet \cdot (NP \backslash S) / NP \cdot \bullet \mathbf{I} \vdash \cdot S \cdot & \downarrow \\
\hline
\cdot NP \cdot \bullet \cdot (NP \backslash S) / NP \cdot \bullet \mathbf{I} \vdash \cdot NP \cdot \backslash \cdot S \cdot & \mathrm{Res}\bullet\backslash \\
\hline
\cdot NP \cdot \bullet \mathbf{I} \bullet \cdot (NP \backslash S) / NP \cdot \bullet \mathbf{I} \vdash \cdot NP \cdot \backslash \cdot S \cdot & \downarrow \\
\hline
\cdot NP \cdot \bullet \cdot NP \cdot \bullet \mathbf{I} \bullet \cdot (NP \backslash S) / NP \cdot \bullet \mathbf{I} \vdash \cdot S \cdot & \mathrm{Res}\backslash\bullet \\
\hline
\cdot NP \cdot \bullet \cdot NP \cdot \bullet \cdot (NP \backslash S) / NP \cdot \bullet \mathbf{I} \vdash \cdot S \cdot & \uparrow \\
\hline
\cdot NP \cdot \bullet \cdot (NP \backslash S) / NP \cdot \bullet \cdot NP \cdot \vdash \cdot S \cdot & \uparrow
\end{array}
$$

Quantifier Raising and NL_{IBC}

$$\text{Structure}^{+} \quad \Gamma \coloneqq \ldots \mid \mathbf{I} \mid \mathbf{B} \mid \mathbf{C}$$

$$\frac{\cdot A \cdot \bullet \mathbf{I} \vdash \Delta}{\cdot A \cdot \vdash \Delta}\,\text{L}\mathbf{I} \qquad \frac{\Gamma \vdash \cdot B \cdot}{\Gamma \bullet \mathbf{I} \vdash \cdot B \cdot}\,\text{R}\mathbf{I}$$

$$\frac{\Gamma_1 \bullet (\Gamma_2 \bullet \Gamma_3) \vdash \Delta}{\Gamma_2 \bullet ((\mathbf{B} \bullet \Gamma_1) \bullet \Gamma_3) \vdash \Delta}\,\mathbf{B} \qquad \frac{(\Gamma_1 \bullet \Gamma_2) \bullet \Gamma_3 \vdash \Delta}{\Gamma_1 \bullet ((\mathbf{C} \bullet \Gamma_2) \bullet \Gamma_3) \vdash \Delta}\,\mathbf{C}$$

Quantifier Raising and NL_{IBC}

$$\cdot NP\cdot \bullet \cdot(NP \backslash S) / NP\cdot \bullet \cdot NP\cdot \vdash \cdot S\cdot$$

$$\text{Res}\bullet\backslash$$

$$\cdot(NP \backslash S) / NP\cdot \bullet \cdot NP\cdot \vdash \cdot NP\cdot \backslash \cdot S\cdot$$

$$\text{Res}\bullet\backslash$$

$$\cdot NP\cdot \vdash \cdot(NP \backslash S) / NP\cdot \backslash \cdot NP\cdot \backslash \cdot S\cdot$$

$$\text{R}\mathbf{I}$$

$$\cdot NP\cdot \bullet \mathbf{I} \vdash \cdot(NP \backslash S) / NP\cdot \backslash \cdot NP\cdot \backslash \cdot S\cdot$$

$$\text{Res}\backslash\bullet$$

$$\cdot(NP \backslash S) / NP\cdot \bullet (\cdot NP\cdot \bullet \mathbf{I}) \vdash \cdot NP\cdot \backslash \cdot S\cdot$$

$$\mathbf{B}$$

$$\cdot NP\cdot \bullet ((\mathbf{B} \bullet \cdot(NP \backslash S) / NP\cdot) \bullet \mathbf{I}) \vdash \cdot NP\cdot \backslash \cdot S\cdot$$

$$\text{Res}\bullet\backslash$$

$$\cdot NP\cdot \bullet (\cdot NP\cdot \bullet ((\mathbf{B} \bullet \cdot(NP \backslash S) / NP\cdot) \bullet \mathbf{I})) \vdash \cdot S\cdot$$

$$\mathbf{B}$$

$$\cdot NP\cdot \bullet ((\mathbf{B} \bullet \cdot NP\cdot) \bullet ((\mathbf{B} \bullet \cdot(NP \backslash S) / NP\cdot) \bullet \mathbf{I})) \vdash \cdot S\cdot$$

$$\text{Res}\bullet\backslash$$

$$((\mathbf{B} \bullet \cdot NP\cdot) \bullet ((\mathbf{B} \bullet \cdot(NP \backslash S) / NP\cdot) \bullet \mathbf{I})) \vdash \cdot NP\cdot \backslash \cdot S\cdot$$

$$\text{R}\backslash$$

$$((\mathbf{B} \bullet \cdot NP\cdot) \bullet ((\mathbf{B} \bullet \cdot(NP \backslash S) / NP\cdot) \bullet \mathbf{I})) \vdash \cdot NP \backslash S\cdot$$

$$\frac{}{\cdot S\cdot \vdash \cdot S\cdot}\ \text{Ax}$$

$$\text{L}/$$

$$\cdot S / (NP \backslash S)\cdot \vdash \cdot S\cdot / ((\mathbf{B} \bullet \cdot NP\cdot) \bullet ((\mathbf{B} \bullet \cdot(NP \backslash S) / NP\cdot) \bullet \mathbf{I}))$$

$$\text{Res}/\bullet$$

$$\cdot S / (NP \backslash S)\cdot \bullet ((\mathbf{B} \bullet \cdot NP\cdot) \bullet ((\mathbf{B} \bullet \cdot(NP \backslash S) / NP\cdot) \bullet \mathbf{I})) \vdash \cdot S\cdot$$

$$\mathbf{B}'$$

$$\cdot NP\cdot \bullet (\cdot S / (NP \backslash S)\cdot \bullet ((\mathbf{B} \bullet \cdot(NP \backslash S) / NP\cdot) \bullet \mathbf{I})) \vdash \cdot S\cdot$$

$$\text{Res}\bullet\backslash$$

$$\cdot S / (NP \backslash S)\cdot \bullet ((\mathbf{B} \bullet \cdot(NP \backslash S) / NP\cdot) \bullet \mathbf{I}) \vdash \cdot NP\cdot \backslash \cdot S\cdot$$

$$\mathbf{B}'$$

$$\cdot(NP \backslash S) / NP\cdot \bullet (\cdot S / (NP \backslash S)\cdot \bullet \mathbf{I}) \vdash \cdot NP\cdot \backslash \cdot S\cdot$$

$$\text{Res}\backslash\bullet$$

$$\cdot S / (NP \backslash S)\cdot \bullet \mathbf{I} \vdash \cdot(NP \backslash S) / NP\cdot \backslash \cdot NP\cdot \backslash \cdot S\cdot$$

$$\text{L}\mathbf{I}$$

$$\cdot S / (NP \backslash S)\cdot \vdash \cdot(NP \backslash S) / NP\cdot \backslash \cdot NP\cdot \backslash \cdot S\cdot$$

$$\text{Res}\bullet\backslash$$

$$\cdot(NP \backslash S) / NP\cdot \bullet \cdot S / (NP \backslash S)\cdot \vdash \cdot NP\cdot \backslash \cdot S\cdot$$

$$\text{Res}\bullet\backslash$$

$$\cdot NP\cdot \bullet \cdot(NP \backslash S) / NP\cdot \bullet \cdot S / (NP \backslash S)\cdot \vdash \cdot S\cdot$$

Quantifier Raising and NL_{IBC}

$$\cdot NP\cdot \bullet \cdot(NP \setminus S) / NP\cdot \bullet \cdot NP\cdot \vdash \cdot S\cdot$$

$$\vdots$$

$$\dfrac{\dfrac{}{\cdot S\cdot \vdash \cdot S\cdot}\text{Ax} \qquad \dfrac{\cdot NP\cdot \bullet ((\mathbf{B} \bullet \cdot NP\cdot) \bullet ((\mathbf{B} \bullet \cdot(NP \setminus S) / NP\cdot) \bullet \mathbf{I})) \vdash \cdot S\cdot}{((\mathbf{B} \bullet \cdot NP\cdot) \bullet ((\mathbf{B} \bullet \cdot(NP \setminus S) / NP\cdot) \bullet \mathbf{I})) \vdash \cdot NP \setminus S\cdot}\text{R}\setminus}{\cdot S / (NP \setminus S)\cdot \bullet ((\mathbf{B} \bullet \cdot NP\cdot) \bullet ((\mathbf{B} \bullet \cdot(NP \setminus S) / NP\cdot) \bullet \mathbf{I})) \vdash \cdot S\cdot}\text{L}/$$

$$\vdots$$

$$\cdot NP\cdot \bullet \cdot(NP \setminus S) / NP\cdot \bullet \cdot S / (NP \setminus S)\cdot \vdash \cdot S\cdot$$

Quantifier Raising and NL_{IBC}

$$\vdots$$
$$\cdot S / (NP \setminus S)\cdot \bullet (\mathbf{B} \bullet \cdot S / (NP \setminus S)\cdot \bullet (\mathbf{C} \bullet (\mathbf{B} \bullet \mathbf{B}) \bullet \mathbf{I}) \bullet (\mathbf{B} \bullet \cdot(NP \setminus S) / NP\cdot) \bullet \mathbf{I} \vdash \cdot S\cdot$$
$$\vdots$$
$$\cdot S / (NP \setminus S)\cdot \bullet (\mathbf{B} \bullet \cdot S / (NP \setminus S)\cdot) \bullet (\mathbf{B} \bullet \cdot(NP \setminus S) / NP\cdot) \bullet \mathbf{I} \vdash \cdot S\cdot$$
$$\vdots$$
$$\cdot S / (NP \setminus S)\cdot \bullet \cdot(NP \setminus S) / NP\cdot \bullet \cdot S / (NP \setminus S)\cdot \vdash \cdot S\cdot$$

Quantifier Raising and $\mathsf{NL}_{\mathsf{IBC}}$

$$\begin{array}{ll} \text{Structure}^{+} & \Gamma \coloneqq \ldots \mid \Gamma_1 \circ \Gamma_2 \mid \mathbf{I} \mid \mathbf{B} \mid \mathbf{C} \\ \text{Structure}^{-} & \Delta \coloneqq \ldots \mid \Delta \mathbin{/\!\!/} \Gamma \mid \Gamma \mathbin{\backslash\!\!\backslash} \Delta \end{array}$$

(copy of rules for $\{\backslash, \bullet, /\}$ for $\{\backslash\!\!\backslash, \circ, /\!\!/\}$)

$$\frac{\cdot A \cdot \circ \mathbf{I} \vdash \Delta}{\cdot A \cdot \vdash \Delta}\,\mathsf{L}\mathbf{I} \qquad \frac{\Gamma \vdash \cdot B \cdot}{\Gamma \circ \mathbf{I} \vdash \cdot B \cdot}\,\mathsf{R}\mathbf{I}$$

$$\frac{\Gamma_1 \bullet (\Gamma_2 \circ \Gamma_3) \vdash \Delta}{\Gamma_2 \circ ((\mathbf{B} \bullet \Gamma_1) \bullet \Gamma_3) \vdash \Delta}\,\mathbf{B} \qquad \frac{(\Gamma_1 \circ \Gamma_2) \bullet \Gamma_3 \vdash \Delta}{\Gamma_1 \circ ((\mathbf{C} \bullet \Gamma_2) \bullet \Gamma_3) \vdash \Delta}\,\mathbf{C}$$

Quantifier Raising and $\mathrm{NL}_{\mathrm{IBC}}$

$$\begin{array}{c} \vdots \\ \cdot(\mathrm{NP} \backslash \mathrm{S}) / \mathrm{NP}\cdot \circ (\mathbf{B} \bullet \cdot\mathrm{NP}\cdot) \bullet ((\mathbf{C} \bullet \mathbf{I}) \bullet \cdot\mathrm{NP}\cdot) \\ \vdots \\ \cdot\mathrm{NP}\cdot \bullet \cdot(\mathrm{NP} \backslash \mathrm{S}) / \mathrm{NP}\cdot \bullet \cdot\mathrm{NP}\cdot \vdash \cdot\mathrm{S}\cdot \end{array}$$

$$\begin{array}{c} \vdots \\ ((\cdot\mathrm{NP}\cdot \circ \mathbf{I}) \circ \mathbf{I}) \bullet \cdot\mathrm{NP} \backslash \mathrm{S}\cdot \bullet \vdash \cdot\mathrm{S}\cdot \\ \vdots \\ (\cdot\mathrm{NP}\cdot \circ \mathbf{I}) \bullet \cdot\mathrm{NP} \backslash \mathrm{S}\cdot \bullet \vdash \cdot\mathrm{S}\cdot \\ \vdots \\ \cdot\mathrm{NP}\cdot \bullet \cdot\mathrm{NP} \backslash \mathrm{S}\cdot \bullet \vdash \cdot\mathrm{S}\cdot \end{array}$$

Quantifier Raising and $\mathsf{NL_{IBC}}$

$$
\begin{array}{rl}
\text{Type} & A, B \coloneqq \ldots \mid \mathbf{Q}(A) \\
\text{Structure}^{+} & \Gamma \coloneqq \ldots \mid \Gamma_1 \circ \Gamma_2 \mid \mathbf{I} \mid \mathbf{B} \mid \mathbf{C} \\
\text{Structure}^{-} & \Delta \coloneqq \ldots \mid \Delta \mathbin{/\!\!/} \Gamma \mid \Gamma \mathbin{\backslash\!\!\backslash} \Delta
\end{array}
$$

(copy of rules for $\{\backslash, \bullet, /\}$ for $\{\backslash\!\!\backslash, \circ, /\!\!/\}$)

$$
\frac{\cdot A \cdot \circ \mathbf{I} \vdash \Delta}{\cdot \mathbf{Q}(A) \cdot \vdash \Delta}\,\mathsf{L}\mathbf{I}
\qquad
\frac{\Gamma \vdash \cdot B \cdot}{\Gamma \circ \mathbf{I} \vdash \cdot \mathbf{Q}(B) \cdot}\,\mathsf{R}\mathbf{I}
\qquad
\frac{\Gamma \vdash \Delta}{\Gamma \circ \mathbf{I} \vdash \Delta}\,\mathbf{I}^{-}
$$

$$
\frac{\Gamma_1 \bullet (\Gamma_2 \circ \Gamma_3) \vdash \Delta}{\Gamma_2 \circ ((\mathbf{B} \bullet \Gamma_1) \bullet \Gamma_3) \vdash \Delta}\,\mathbf{B}
\qquad
\frac{(\Gamma_1 \circ \Gamma_2) \bullet \Gamma_3 \vdash \Delta}{\Gamma_1 \circ ((\mathbf{C} \bullet \Gamma_2) \bullet \Gamma_3) \vdash \Delta}\,\mathbf{C}
$$

Quantifier Raising and NL_{IBC}

$$\cdot NP\cdot \bullet \cdot (NP \setminus S) / NP\cdot \bullet \cdot NP\cdot \vdash \cdot S\cdot$$

$$\vdots$$

$$\dfrac{\dfrac{}{\cdot S\cdot \vdash \cdot S\cdot}\,\text{Ax} \qquad \dfrac{\cdot NP\cdot \circ ((\mathbf{B} \bullet \cdot NP\cdot) \bullet ((\mathbf{B} \bullet \cdot (NP \setminus S) / NP\cdot) \bullet \mathbf{I})) \vdash \cdot S\cdot}{((\mathbf{B} \bullet \cdot NP\cdot) \bullet ((\mathbf{B} \bullet \cdot (NP \setminus S) / NP\cdot) \bullet \mathbf{I})) \vdash \cdot NP \mathbin{\setminus\!\!\setminus} S\cdot}\,\text{R}\setminus}{\cdot S \mathbin{/\!\!/} (NP \mathbin{\setminus\!\!\setminus} S)\cdot \circ ((\mathbf{B} \bullet \cdot NP\cdot) \bullet ((\mathbf{B} \bullet \cdot (NP \mathbin{\setminus\!\!\setminus} S) \mathbin{/\!\!/} NP\cdot) \bullet \mathbf{I})) \vdash \cdot S\cdot}\,\text{L}/$$

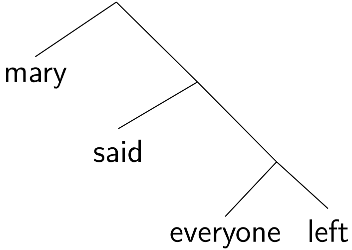
$$\vdots$$

$$\cdot NP\cdot \bullet \cdot (NP \setminus S) / NP\cdot \bullet \cdot \mathbf{Q}(S \mathbin{/\!\!/} (NP \mathbin{\setminus\!\!\setminus} S))\cdot \vdash \cdot S\cdot$$

Scope Islands

"Mary <u>said</u> everyone left."

Given that the parse tree is:



And the denotation is:

$$\textbf{said}(\text{mary}, \forall x.\textbf{left}(x))$$

And definitely isn't:

$$\forall x.\textbf{said}(\text{mary}, \textbf{left}(x))$$

Quantifier Raising and Scope Islands

$$\begin{array}{rr} \text{Type} & A, B \coloneqq \ldots \mid \Diamond A \mid \Box A \\ \text{Structure}^{+} & \Gamma \coloneqq \ldots \mid \langle \Gamma \rangle \\ \text{Structure}^{-} & \Delta \coloneqq \ldots \mid [\Delta] \end{array}$$

$$\frac{\langle \cdot A \cdot \rangle \vdash \Delta}{\cdot \Diamond A \cdot \vdash \Delta}\,\mathsf{L}\Diamond \qquad \frac{\Gamma \vdash \cdot B \cdot}{\langle \Gamma \rangle \vdash \cdot \Diamond B \cdot}\,\mathsf{R}\Diamond$$

$$\frac{\cdot A \cdot \vdash \Delta}{\cdot \Box A \cdot \vdash [\Delta]}\,\mathsf{L}\Box \qquad \frac{\Gamma \vdash [\cdot B \cdot]}{\Gamma \vdash \cdot \Box B \cdot}\,\mathsf{R}\Box$$

$$\frac{\Gamma \vdash [\Delta]}{\overline{\langle \Gamma \rangle \vdash \Delta}}\,\mathsf{Res}\Box\Diamond$$

Quantifier Raising and Scope Islands

"Mary <u>said</u> everyone left."

$$\cdot\mathrm{NP}\cdot \bullet \cdot(\mathrm{NP} \setminus \mathrm{S}) \,/\, \mathrm{S}\cdot \bullet \cdot\mathbf{Q}(\mathrm{S} \;/\!\!/\; (\mathrm{NP} \setminus\!\!\setminus \mathrm{S}))\cdot \bullet \cdot\mathrm{NP} \setminus \mathrm{S}\cdot \vdash \cdot\mathrm{S}\cdot$$

Quantifier Raising and Scope Islands

"Mary <u>said</u> everyone left."

$$\dfrac{\dfrac{\dfrac{\vdots}{\cdot\mathsf{NP}\cdot \bullet \cdot\mathsf{NP} \setminus \mathsf{S}\cdot \vdash \cdot\mathsf{S}\cdot}}{\vdots}}{\dfrac{\cdot\mathbf{Q}(\mathsf{S} \mathbin{/\!\!/} (\mathsf{NP} \mathbin{\backslash\!\!\backslash} \mathsf{S}))\cdot \bullet \cdot\mathsf{NP} \setminus \mathsf{S}\cdot \vdash \cdot\mathsf{S}\cdot}{\langle\cdot\mathbf{Q}(\mathsf{S} \mathbin{/\!\!/} (\mathsf{NP} \mathbin{\backslash\!\!\backslash} \mathsf{S}))\cdot \bullet \cdot\mathsf{NP} \setminus \mathsf{S}\cdot\rangle \vdash \cdot\Diamond\mathsf{S}\cdot}\,\mathsf{R}\Diamond}$$

$$\vdots$$

$$\cdot\mathsf{NP}\cdot \bullet \cdot(\mathsf{NP} \setminus \mathsf{S}) / \Diamond\mathsf{S}\cdot \bullet \langle\cdot\mathbf{Q}(\mathsf{S} \mathbin{/\!\!/} (\mathsf{NP} \mathbin{\backslash\!\!\backslash} \mathsf{S}))\cdot \bullet \cdot\mathsf{NP} \setminus \mathsf{S}\cdot\rangle \vdash \cdot\mathsf{S}\cdot$$

Conclusion

We have:

- set up a logical calculus;
- has a decidable proof search;
- which can deal with:
  - adjacent composition;
  - quantifier raising;
  - scope islands;

  - infixation – i.e. moving up and staying there;
  - extraction – i.e. moving down and staying there.

Future Work

Mary:NP see:TV.PAST fox:NP.PL

$\downarrow$

$\boxed{\text{Syntactic}}$

$\downarrow$

Mary:NP [see:TV.PAST fox:NP.PL]

$\downarrow$

$\boxed{\text{Semantic}}$

$\downarrow$

$$\exists p.\forall x.p(x) \supset (\mathbf{fox}(x) \wedge \mathbf{past}(\mathbf{see}(\text{mary}, x)))$$

Future Work

Forward-Chaining Proof Search:

- generate all possible sentences;
- filter on the sentences with the right word order.

Future Work

Forward-Chaining Proof Search:

- generate all possible sentences;
- filter on the sentences with the right word order.

Weak vs. Strong quantifiers:

- existential quantifier can sometimes move out of scope islands where universal cannot;
- boxes might be useful, since they can cancel out diamonds i.e. using $\mathbf{Q}(\mathsf{S} \,/\, (\Box\mathsf{NP} \setminus \mathsf{S}))$;
- further research is needed.

Conclusion

We have:

- set up a logical calculus;
- has a decidable proof search;
- which can deal with:
  - adjacent composition;
  - quantifier raising;
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  - infixation – i.e. moving up and staying there;
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Bonus Slides

L$\mathbin{/\!\!/}\downarrow$ and R$\mathbin{\backslash\!\!\backslash}\uparrow$ as derivable rules

$$\cdot \mathsf{NP}\cdot \bullet \cdot(\mathsf{NP} \backslash \mathsf{S}) / \mathsf{NP}\cdot \bullet \cdot \mathsf{NP}\cdot \vdash \cdot \mathsf{S}\cdot$$

$$\vdots$$

$$\dfrac{\dfrac{}{\cdot \mathsf{S}\cdot \vdash \cdot \mathsf{S}\cdot}\,\mathrm{Ax} \qquad \dfrac{\cdot \mathsf{NP}\cdot \circ ((\mathbf{B} \bullet \cdot \mathsf{NP}\cdot) \bullet ((\mathbf{B} \bullet \cdot(\mathsf{NP} \backslash \mathsf{S}) / \mathsf{NP}\cdot) \bullet \mathbf{I})) \vdash \cdot \mathsf{S}\cdot}{((\mathbf{B} \bullet \cdot \mathsf{NP}\cdot) \bullet ((\mathbf{B} \bullet \cdot(\mathsf{NP} \backslash \mathsf{S}) / \mathsf{NP}\cdot) \bullet \mathbf{I})) \vdash \cdot \mathsf{NP} \mathbin{\backslash\!\!\backslash} \mathsf{S}\cdot}\,\mathrm{R}\backslash\!\!\backslash}{\cdot \mathsf{S} \mathbin{/\!\!/} (\mathsf{NP} \mathbin{\backslash\!\!\backslash} \mathsf{S})\cdot \circ ((\mathbf{B} \bullet \cdot \mathsf{NP}\cdot) \bullet ((\mathbf{B} \bullet \cdot(\mathsf{NP} \backslash \mathsf{S}) / \mathsf{NP}\cdot) \bullet \mathbf{I})) \vdash \cdot \mathsf{S}\cdot}\,\mathrm{L}/\!\!/$$

$$\vdots$$

$$\cdot \mathsf{NP}\cdot \bullet \cdot(\mathsf{NP} \backslash \mathsf{S}) / \mathsf{NP}\cdot \bullet \cdot \mathbf{Q}(\mathsf{S} \mathbin{/\!\!/} (\mathsf{NP} \mathbin{\backslash\!\!\backslash} \mathsf{S}))\cdot \vdash \cdot \mathsf{S}\cdot$$

L $\sslash$$\downarrow$ and R$\bbslash$$\uparrow$ as derivable rules

$$
\dfrac{
  \dfrac{}{\cdot\mathrm{S}\cdot \vdash \cdot\mathrm{S}\cdot}\,\mathrm{Ax}
  \qquad
  \dfrac{
    \dfrac{
      \cdot\mathrm{NP}\cdot \bullet \cdot(\mathrm{NP} \backslash \mathrm{S}) / \mathrm{NP}\cdot \bullet \cdot\mathrm{NP}\cdot \vdash \cdot\mathrm{S}\cdot
    }{
      \cdot\mathrm{NP}\cdot \circ ((\mathbf{B} \bullet \cdot\mathrm{NP}\cdot) \bullet ((\mathbf{B} \bullet \cdot(\mathrm{NP} \backslash \mathrm{S}) / \mathrm{NP}\cdot) \bullet \mathbf{I})) \vdash \cdot\mathrm{S}\cdot
    }\,\uparrow
  }{
    ((\mathbf{B} \bullet \cdot\mathrm{NP}\cdot) \bullet ((\mathbf{B} \bullet \cdot(\mathrm{NP} \backslash \mathrm{S}) / \mathrm{NP}\cdot) \bullet \mathbf{I})) \vdash \cdot\mathrm{NP} \bbslash \mathrm{S}\cdot
  }\,\mathrm{R}\bbslash
}{
  \dfrac{
    \cdot\mathrm{S} \sslash (\mathrm{NP} \bbslash \mathrm{S})\cdot \circ ((\mathbf{B} \bullet \cdot\mathrm{NP}\cdot) \bullet ((\mathbf{B} \bullet \cdot(\mathrm{NP} \backslash \mathrm{S}) / \mathrm{NP}\cdot) \bullet \mathbf{I})) \vdash \cdot\mathrm{S}\cdot
  }{
    \cdot\mathrm{NP}\cdot \bullet \cdot(\mathrm{NP} \backslash \mathrm{S}) / \mathrm{NP}\cdot \bullet \cdot\mathbf{Q}(\mathrm{S} \sslash (\mathrm{NP} \bbslash \mathrm{S}))\cdot \vdash \cdot\mathrm{S}\cdot
  }\,\downarrow
}\,\mathrm{L}\sslash
$$

L $\varoslash\!\downarrow$ and R$\varobslash\!\uparrow$ as derivable rules

$$\dfrac{\dfrac{}{\cdot S\cdot \vdash \cdot {\color{red}S}\cdot}\,\text{Ax} \qquad \dfrac{\cdot NP\cdot \bullet \cdot (NP \setminus S) \,/\, NP\cdot \bullet \cdot {\color{red}NP}\cdot \vdash \cdot {\color{red}S}\cdot}{((\mathbf{B} \bullet \cdot NP\cdot) \bullet ((\mathbf{B} \bullet \cdot (NP \setminus S) \,/\, NP\cdot) \bullet \mathbf{I})) \vdash \cdot {\color{red}NP \varobslash S}\cdot}\,\text{R}\varobslash\!\uparrow}{\cdot NP\cdot \bullet \cdot (NP \setminus S) \,/\, NP\cdot \bullet \cdot {\color{red}\mathbf{Q}(S \varoslash (NP \varobslash S))}\cdot \vdash \cdot S\cdot}\,\text{L}\varoslash\!\downarrow$$

L$\mathbin{/\!\!/}\!\downarrow$ and R$\mathbin{\backslash\!\!\backslash}\!\uparrow$ as derivable rules

$$\text{Context } \Sigma \coloneqq \square \mid \Sigma \bullet \Delta \mid \Gamma \bullet \Sigma$$

$$\begin{array}{rlll}
\square[\Gamma] & \mapsto \Gamma & \mathrm{Trace}(\square) & \mapsto \mathbf{I} \\
(\Sigma \bullet \Delta)[\Gamma] & \mapsto (\Sigma[\Gamma] \bullet \Delta) & \mathrm{Trace}(\Sigma \bullet \Delta) & \mapsto ((\mathbf{C} \bullet \Sigma[\Gamma]) \bullet \Delta) \\
(\Delta \bullet \Sigma)[\Gamma] & \mapsto (\Delta \bullet \Sigma[\Gamma]) & \mathrm{Trace}(\Delta \bullet \Sigma) & \mapsto ((\mathbf{B} \bullet \Delta) \bullet \Sigma[\Gamma])
\end{array}$$

$$\frac{\cdot C \cdot \vdash \Delta \qquad \mathrm{Trace}(\Sigma) \vdash \cdot B \cdot}{\Sigma[\cdot \mathbf{Q}(C \mathbin{/\!\!/} B) \cdot] \vdash \Delta}\,\mathrm{L}\mathbin{/\!\!/}\!\downarrow \qquad \frac{\Sigma[\cdot A \cdot] \vdash \cdot B \cdot}{\mathrm{Trace}(\Sigma) \vdash \cdot A \setminus B \cdot}\,\mathrm{R}\mathbin{\backslash\!\!\backslash}\!\uparrow$$